UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 7
Fall 2021

1. Recurrence and Transience of Random Walks

"A drunk man will eventually find his way home but a drunk bird may get lost forever."

Consider the symmetric random walk $S_n = X_1 + \cdots + X_n$ in dimension d, where we start at the origin, and with uniform probability we jump to an adjacent point on the d-dimensional lattice $\mathbb{Z}^d$. That is, $X_i \overset{iid}{\sim} \text{Uniform}\{\pm e_1, \ldots, \pm e_d\}$, where $\{e_1, \ldots, e_d\}$ are the unit coordinate vectors in $\mathbb{R}^d$.

(a) Show that if $\sum_{n=0}^{\infty} P(S_n = 0) = \infty$ then the random walk is recurrent. *Hint:* Let N be the number of times the random walk visits the origin. It may help to notice that $\mathbb{E}[N] = \infty$ is equivalent to recurrence of the random walk.

(b) Use this to show that the random walk for $d = 1$ is recurrent. You may use Stirling's approximation:
$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

(c) Use part (b) to show that the random walk for $d = 2$ is recurrent.

(d) (Optional) Show that the random walk for $d = 3$ is transient.

(e) Use part (d) to show that the random walk for any $d > 3$ is also transient.

2. Arrival Times of a Poisson Process

Consider a Poisson process $(N_t, t \geq 0)$ with rate $\lambda = 1$. For $i \in \mathbb{Z}_{>0}$, let T_i be a random variable which is equal to the time of the i-th arrival.

(a) Find $\mathbb{E}[T_3 \mid N_1 = 2]$.

(b) Given $T_3 = s$, where $s > 0$, find the joint distribution of T_1 and T_2.

(c) Find $\mathbb{E}[T_2 \mid T_3 = s]$.

3. Illegal U-Turns

Each morning, as you pull out of your driveway, you would like to make a U-turn rather than drive around the block. Unfortunately, U-turns are illegal and police cars drive by according to a Poisson process with rate λ. You decide to make a U-turn once you see that the road has been clear of police cars for $\tau > 0$ units of time. Let N be the number of police cars you see before you make a U-turn.

(a) Find $\mathbb{E}[N]$.

(b) Let n be a positive integer ≥ 2. Find the conditional expectation of the time elapsed between police cars $n-1$ and n, given that $N \geq n$.

(c) Find the expected time that you wait until you make a U-turn.