EECS 126 Probability and Random Processes — University of California, Berkeley: Fall 2017

Abhay Parekh and Jean Walrand — October 26, 2017

Midterm Exam

Last Name	First Name	SID

Rules.

- You have 80 minutes (12:40pm - 2:00pm) to complete this exam.
- The maximum you can score is 120.
- The exam is not open book, but you are allowed one side of a sheet of handwritten notes; calculators will be allowed. No phones.
- No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.

Please read the following remarks carefully.

- Show all work to get any partial credit.
- Take into account the points that may be earned for each problem when splitting your time between the problems.

Problem	points earned	out of
Problem 1		40
Problem 2		20 + 5
Problem 3		20
Problem 4		20
Problem 5		20
Problem 6		20
Total		140 + 5 (Bonus)

Problem 1 [40]

(a) [15] For the Markov chain X_n with the state transition diagram shown in Figure 1, let $T_3 = \min\{n \geq 0 \mid X_n = 3\}$. Assume that X_0 is uniformly distributed in $\{0, 1, 2, 3\}$. Find $E(T_3)$.

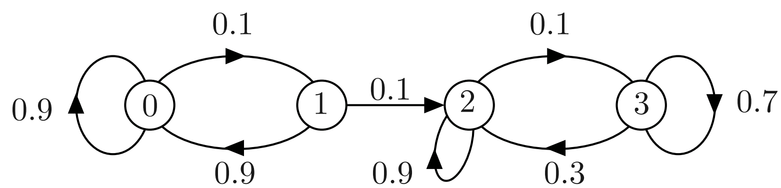

Figure 1: State transition diagram for Problem 1.

Let $\beta(i) = E[T_3 \mid X_0 = i]$. Then

$$\begin{aligned}
\beta(0) &= 1 + 0.9\beta(0) + 0.1\beta(1) \\
\beta(1) &= 1 + 0.9\beta(0) + 0.1\beta(2) \\
\beta(2) &= 1 + 0.9\beta(2).
\end{aligned}$$

The last equation gives $\beta(2) = 10$. The first gives $\beta(0) = 10 + \beta(1)$. The second then gives $\beta(1) = 1 + 0.9\beta(0) + 1 = 2 + 0.9[10 + \beta(1)] = 11 + 0.9\beta(1)$. Hence, $\beta(1) = 110$, so that $\beta(0) = 120$.

Finally, $E(T_3) = 0.25(\beta(0) + \beta(1) + \beta(3) + 0) = 0.25(120 + 110 + 10) = 60$.

(b) [5] What are all the invariant distributions of the Markov chain shown in Figure 1?

It is unique and given by $\begin{bmatrix} 0 & 0 & 0.75 & 0.25 \end{bmatrix}$.

(c) [10] For the same Markov chain, let $T = \max\{n \geq 0 \mid X_n \leq 1\}$. Find $E[T \mid X_0 = 0]$.

One has $T = T_2 - 1$ where $T_2 = \min\{n \geq 0 \mid X_n = 2\}$. Hence, $E[T \mid X_0 = 0] = \beta(0) - 1$ where $\beta(i) = E[T_2 \mid X_0 = i]$.

Now,

$$\begin{aligned}
\beta(0) &= 1 + 0.9\beta(0) + 0.1\beta(1) \\
\beta(1) &= 1 + 0.9\beta(0).
\end{aligned}$$

Hence, $\beta(0) = 10 + \beta(1) = 11 + 0.9\beta(0)$. Thus, $\beta(0) = 110 = E[T_2 \mid X_0 = 0]$, so that $E[T \mid X_0 = 0] = 109$.

(d) [5] For the same Markov chain, what are the long term fractions of time that X_n is in the four different states?

They are $\begin{bmatrix} 0 & 0 & 0.75 & 0.25 \end{bmatrix}$.

(e) [5] For the same Markov chain, assume that $\pi_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$. What can you say about π_n as $n \to \infty$?

One has $\pi_n \to \begin{bmatrix} 0 & 0 & 0.75 & 0.25 \end{bmatrix}$.

Problem 2 [20 + 5 (bonus)] You have 10 quarters in your left pocket and 10 quarters in your right pocket. At each step, you choose one of the two pockets, with equal probabilities, and you remove a quarter from that pocket and put it in the other pocket. What is the average number of steps until you reach into an empty pocket?

(a) [**10**] Formulate the problem as a Markov chain hitting time.

Let X_n be the number of quarters in the left pocket at step n, for $n \geq 0$. Let's add state -1 to mean that you reached in the empty left pocket and state 21 to mean that you reached in the empty right pocket. The transition probabilities are $P(n, n+1) = 0.5$ and $P(n+1, n) = 0.5$. Let T be the first time the Markov chain hits -1 or 21. The problem is to find $E[T \mid X_0 = 10]$.

(b) [**10**] Writing down the first step equations.

The first step equations for $\beta(k) = E[T \mid X_0 = k]$ are

$$\beta(k) = 1 + 0.5\beta(k-1) + 0.5\beta(k+1), \; k = 0, \ldots, 20 \tag{1}$$
$$\beta(-1) = 0 \tag{2}$$
$$\beta(21) = 0. \tag{3}$$

(c) [**5**] **Bonus** Solve the equations.

If we try $\beta(k) = a + bk + ck^2$ (I know...this is why you get bonus points), Equation 1 implies that

$$\begin{aligned} a + bk + ck^2 &= 1 + 0.5\big(a + bk - b + c(k-1)^2\big) + 0.5\big(a + bk + b + c(k+1)^2\big) \\ &= 1 + a + bk + ck^2 + c. \end{aligned}$$

This requires $c = -1$.

Equation 2 then implies that $0 = a - b + c$. Hence, $a - b = -c = 1$. Also, Equation 3 implies that $0 = a + 21b + 441c = a + 21b - 441$, so that $a + 21b = 441$. Subtracting these identities gives $22b = 440$, so that $b = 20$ and $a = 21$.

We conclude that $\beta(k) = 21 + 20k - k^2$. In particular, $\beta(10) = 21 + 200 - 100 = 121$.

Alternatively, by symmetry $\beta(10 - k) = \beta(10 + k)$ for $k = 0, \ldots, 11$. Thus, the equation for $\beta(10)$ gives $\beta(10) = \beta(9) + 1$. The equation for $\beta(9)$ gives $\beta(9) = 0.5\beta(10) + 0.5\beta(8) + 1$, which, combined with the previous equation gives $\beta(8) = \beta(10) - 2^2$, and in general, $\beta(10 - i) = \beta(10) - i^2$ for $i = 0, \ldots, 11$. Then, $\beta(-1) = 0$ implies $\beta(10) = 121$.

Problem 3 [20] Consider the hidden Markov chain $\text{HMC}(\pi_0, P, Q)$ with $\pi_0 = [0.4, 0.6]$ and

$$P = \begin{bmatrix} 0.4 & 0.6 \\ 0.5 & 0.5 \end{bmatrix}, \text{ and } Q = \begin{bmatrix} 0.5 & 0.5 \\ 0.6 & 0.4 \end{bmatrix}.$$

Recall that Q is the emission matrix, so $Q(x, y) = P[Y_0 = y \mid X_0 = x]$ for all x, y. Find $\text{MAP}[X_0, X_1 \mid Y_0 = 0, Y_1 = 1]$.

(a) **[5]** Explain your approach clearly and concisely.

Since maximizing $P[X_0 = a, X_1 = b \mid Y_0 = 0, Y_1 = 1]$ is equivalent to maximizing $g(a, b) := P[X_0 = a, X_1 = b, Y_0 = 0, Y_1 = 1] = \pi_0[a]P(a, b)Q(a, 0)Q(b, 1)$, we calculate the latter for $(a, b) \in \{0, 1\}^2$ and we find the maximizing pair (a, b).

(b) **[10]** Show your calculations clearly.

$$\begin{aligned} g(0,0) &= 0.4 \times 0.4 \times 0.5 \times 0.5 = 0.04 \\ g(0,1) &= 0.4 \times 0.6 \times 0.5 \times 0.4 = 0.048 \\ g(1,0) &= 0.6 \times 0.5 \times 0.6 \times 0.5 = 0.09 \\ g(1,1) &= 0.6 \times 0.5 \times 0.6 \times 0.4 = 0.072. \end{aligned}$$

(c) **[5]** State your result.

We conclude that $\text{MAP}[X_0, X_1 \mid Y_0 = 0, Y_1 = 1] = (1, 0)$.

Problem 4 [20] Let X_n be a Markov chain on $\{0, 1\}$ with $P(X_0 = 0) = 0.5$ and $P(0, 1) = P(1, 0) = 0.5$. Also, let Y_n be an independent Markov chain on $\{0, 1\}$ with $P(Y_0 = 0) = 0.5$ and $P(0, 1) = P(1, 0) = 0.01$. Finally, let $Z_n = X_n + Y_n$ for $n \geq 0$. Prove or disprove that $\{Z_n, n \geq 0\}$ is a Markov chain.

(a) **[10]** Explain your approach clearly and concisely.

We will test whether $P[Z_2 \mid Z_1, Z_0]$ depends on Z_0 for some value of Z_1. If it does, then this proves that $\{Z_n, n \geq 0\}$ is not a Markov chain. Specifically, we will show that $P[Z_2 = 0 \mid Z_1 = 1, Z_0 = 0] \neq P[Z_2 = 0 \mid Z_1 = 1, Z_0 = 2]$.

(b) **[10]** Show your calculations clearly.

One has

$$\begin{aligned} P[Z_2 = 0 \mid Z_1 = 1, Z_0 = 0] &= \frac{P[Z_2 = 0, Z_1 = 1 \mid Z_0 = 0]}{P[Z_1 = 1 \mid Z_0 = 0]} \\ &= \frac{0.5 \times 0.99 \times 0.5 \times 0.99 + 0.5 \times 0.01 \times 0.5 \times 0.01}{0.5 \times 0.01 + 0.5 \times 0.99} \\ &\approx 0.5 \times 0.99. \end{aligned}$$

Also,

$$\begin{aligned} P[Z_2 = 0 \mid Z_1 = 1, Z_0 = 2] &= \frac{P[Z_2 = 0, Z_1 = 1 \mid Z_0 = 2]}{P[Z_1 = 1 \mid Z_0 = 2]} \\ &= \frac{0.5 \times 0.99 \times 0.5 \times 0.01 + 0.5 \times 0.01 \times 0.5 \times 0.99}{0.5 \times 0.01 + 0.5 \times 0.99} \\ &\approx 0.5 \times 0.02. \end{aligned}$$

The intuition is that if $Z_0 = 0$ and $Z_1 = 1$, it is very likely that $X_1 = 1$ and $Y_1 = 0$, so that the probability that $Z_2 = 0$ is close to the probability 0.5 that $X_2 = 0$ given that $X_1 = 1$. On the other hand, if $Z_0 = 2$ and $Z_1 = 1$, it is very likely that $Y_1 = 1$ and $X_1 = 0$, so that the probability that $Z_2 = 0$ is close to the probability 0.01×0.5 that Y_2 switches from 1 to 0 and X_2 stays in 0.

Problem 5 **[20]** Customers enter a burger chain restaurant following a Poisson process with rate 100. Every customer gets their burger "Animal Style" with probability p. We know that 500 customers arrive in the first 5 hours of the day.

In answering the following questions state your reasoning as clearly as you can.

(a) **[10]** Find the probability that n customers ordered their burger "Animal Style" in the first 5 hours.

Each of the 500 customers elects to get their burger independently with probability p. So the PMF of the number of customers who do this is $\text{Binomial}(500, p)$, i.e.,

$$P\big(N_A(5) = n \bigm| N(5) = 500\big) = \binom{500}{n} p^n (1-p)^{500-n}.$$

Alternatively, observe that the "Animal Style" customers form a PP, A, with rate $100p$ and the other customers form an independent PP, B, with rate $100(1-p)$. Thus

$$\begin{aligned} P\big(N_A(5) = n \bigm| N(5) = 500\big) &= \frac{P(N_A(5) = n) P(N_B(5) = 500 - n)}{P(N(5) = 500)} \\ &= \frac{e^{-500p}(500p)^n}{n!} \frac{e^{-500(1-p)}(500(1-p))^{500-n}}{(500-n)!} \frac{500!}{e^{-500} 500^{500}} \\ &= \binom{500}{n} \frac{(500p)^n (500(1-p))^{500-n}}{500^{500}} = \binom{500}{n} p^n (1-p)^{500-n}. \end{aligned}$$

(b) [10] Find the probability that n customers ordered their burger animal style in the first 2 hours.

Since we are given then $N(5) = 500$, We know that the unordered arrival times of these 500 customers behave as iid uniform random variables over $[0, 5]$. The probability that any one of them is ≤ 2 is 0.4. Further, the probability that any of these customers arrived in $[0, 2]$ and ordered their burgers animal style is $0.4p$. Thus the required probability is simply

$$\binom{500}{n}(0.4p)^n(1-0.4p)^{500-n}.$$

Problem 6 [20] Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. Bernoulli random variables with mean p. Let also $Y_n = \sum_{m=1}^{n} X_m$ for $n \geq 1$. Find $P[Y_7 = 1 \mid Y_{10} = 1]$.

(a) [10] Explain your approach clearly and concisely.

We will use symmetry that says that all the sequences of 10 Bernoulli random variables with one 1 and nine 0s have the same probability.

(b) [10] Show your calculations.

There are 7 sequences with $Y_7 = 1$ among the 10 equally likely sequences with $Y_{10} = 1$. Thus,

$$P[Y_7 = 1 \mid Y_{10} = 1] = 0.7.$$