UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 12
Fall 2021

1. **Flipping Coins and Hypothesizing**

   You flip a coin until you see heads. Let

$$X = \begin{cases} 1 & \text{if the bias of the coin is } q > p. \\ 0 & \text{if the bias of the coin is } p. \end{cases}$$

   Find a decision rule $\hat{X}(Y)$ that maximizes $P[\hat{X} = 1 \mid X = 1]$ subject to $P[\hat{X} = 1 \mid X = 0] \leq \beta$ for $\beta \in [0, 1]$. Remember to calculate the randomization constant γ.

2. **Gaussian Hypothesis Testing**

   Consider a hypothesis testing problem that if $X = 0$, you observe a sample of $\mathcal{N}(\mu_0, \sigma^2)$, and if $X = 1$, you observe a sample of $\mathcal{N}(\mu_1, \sigma^2)$, where $\mu_0, \mu_1 \in \mathbb{R}$, $\sigma^2 > 0$. Find the Neyman-Pearson test for false alarm $\alpha \in (0, 1)$, that is, $P(\hat{X} = 1 \mid X = 0) \leq \alpha$.

3. **BSC Hypothesis Testing**

   Consider a BSC with some error probability $\epsilon \in [0.1, 0.5)$. Given n inputs and outputs (x_i, y_i) of the BSC, solve a hypothesis problem to detect that $\epsilon > 0.1$ with a probability of false alarm at most equal to 0.05. Assume that n is very large and use the CLT.

   Hint: The null hypothesis is $\epsilon = 0.1$. The alternate hypothesis is $\epsilon > 0.1$, which is a **composite hypothesis** (this means that under the alternate hypothesis, the probability distribution of the observation is not completely determined; compare this to a **simple hypothesis** such as $\epsilon = 0.3$, which *does* completely determine the probability distribution of the observation). The Neyman-Pearson Lemma we learned in class applies for the case of a simple null hypothesis and a simple alternate hypothesis, so it does not directly apply here.

   To fix this, fix some specific $\epsilon' > 0.1$ and use the Neyman-Pearson Lemma to find the optimal hypothesis test for the hypotheses $\epsilon = 0.1$ vs. $\epsilon = \epsilon'$. Then, argue that the optimal decision rule does not depend on the specific choice of ϵ'; thus, the decision rule you derive will be *simultaneously* optimal for testing $\epsilon = 0.1$ vs. $\epsilon = \epsilon'$ for all $\epsilon' > 0.1$.

4. **Basic Properties of Jointly Gaussian Random Variables**

   Let $(X_1, \ldots, X_n)$ be a collection of jointly Gaussian random variables. Their joint density is given by (for $x \in \mathbb{R}^n$)

$$f(x) = \frac{1}{\sqrt{(2\pi)^n \det(C)}} \exp\left(-\frac{1}{2}(x - \mu)^T C^{-1}(x - \mu)\right),$$

   where μ is the mean vector and C is the covariance matrix.

   (a) Show that $X_1, \ldots, X_n$ are independent if and only if they are pairwise uncorrelated.

(b) Show that any linear combination of these random variables will also be a Gaussian random variable.

5. **Independent Gaussians**

Let $X = (X, Y)$ be a jointly Gaussian random vector with mean vector $[0, 0]$ and covariance matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Find a 2×2 matrix U such that $UX = (X', Y')$ where X' and Y' are independent.