UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: PROBABILITY AND RANDOM PROCESSES

Problem Set 13
Fall 2021

1. **Projections**

   The following exercises are from the note on the Hilbert space of random variables. See the notes for some hints.

   (a) Let $\mathcal{H} := \{X : X \text{ is a real-valued random variable with } \mathbb{E}[X^2] < \infty\}$. Prove that $\langle X, Y \rangle := \mathbb{E}[XY]$ makes $\mathcal{H}$ into a real inner product space. [1]

   (b) Let U be a subspace of a real inner product space V and let P be the projection map onto U. Prove that P is a linear transformation.

   (c) Suppose that U is finite-dimensional, $n := \dim U$, with basis $\{v_i\}_{i=1}^n$. Suppose that the basis is orthonormal. Show that $Py = \sum_{i=1}^n \langle y, v_i \rangle v_i$. (Note: If we take $U = \mathbb{R}^n$ with the standard inner product, then P can be represented as a matrix in the form $P = \sum_{i=1}^n v_i v_i^\mathsf{T}$.)

2. **Exam Difficulties**

   The difficulty of an EECS 126 exam, Θ, is uniformly distributed on $[0, 100]$ (i.e. continuous distribution, not discrete), and Alice gets a score X that is uniformly distributed on $[0, \Theta]$. Alice gets her score back and wants to estimate the difficulty of the exam.

   (a) What is the MLE of Θ? What is the MAP of Θ?

   (b) What is the LLSE for Θ?

3. **Jointly Gaussian Decomposition**

   Let U and V be jointly Gaussian random variables with means $\mu_U = 1$, $\mu_V = 4$, respectively, with variances $\sigma_U^2 = 2.5$, $\sigma_V^2 = 2$, respectively, and with covariance $\rho = 1$. Can we write U as $U = aV + Z$, where a is a scalar and Z is independent of V? If you think we can, find the value of a and the distribution of Z; otherwise please explain the reason.

4. **Photodetector LLSE**

   Consider a photodetector in an optical communications system that counts the number of photons arriving during a certain interval. A user conveys information by switching a photon transmitter on or off. Assume that the probability of the transmitter being on is p. If the transmitter is on, the number of photons transmitted over the interval of interest is a Poisson random variable Θ with mean λ, and if it is off, the number of photons transmitted is 0. Unfortunately, regardless of whether or not the transmitter is on or off, photons may be detected due to "shot noise". The number N of detected shot noise photons is a Poisson random variable N with mean μ, independent of the transmitted photons. Let T be the number of transmitted photons and D be the number of detected photons. Find $L[T \mid D]$.

[1] To be perfectly correct, it is possible for $X \neq 0$ but $\mathbb{E}[X^2] = 0$; this occurs if $X = 0$ with probability 1. To fix this, we need to define two random variables X and Y to be *equal* if $P(X = Y) = 1$. In other words, we consider *equivalence classes* of random variables, defined by the relation $\overset{a.s.}{=}$. With this definition, then if $X \neq 0$ we do indeed have $\mathbb{E}[X^2] > 0$.