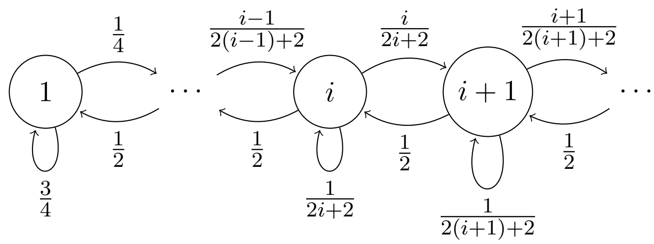
UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 7
Fall 2021

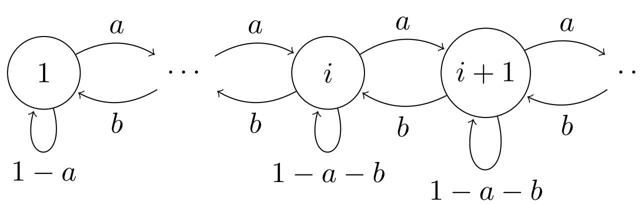
1. Markov Chains with Countably Infinite State Space

(a) Consider a Markov chain with state space $\mathbb{Z}_{>0}$ and transition probability graph:



Show that this Markov chain has no stationary distribution.

(b) Consider a Markov chain with state space $\mathbb{Z}_{>0}$ and transition probability graph:



Assume that $0 < a < b$ and $0 < a + b \leq 1$. Show that the probability distribution given by

$$\pi(i) = \left(\frac{a}{b}\right)^{i-1}\left(1 - \frac{a}{b}\right), \text{ for } i \in \mathbb{Z}_{>0},$$

is a stationary distribution of this Markov chain.

2. Poisson Branching

Consider a branching process such that at generation n, each individual in the population survives until generation $n+1$ with probability $0 < p < 1$, independently, and a Poisson number (with parameter λ) of immigrants enters the population. Let X_n denote the number of people in the population at generation n.

(a) Suppose that at generation 0, the number of people in the population is a Poisson random variable with parameter λ_0. What is the distribution at generation 1? What is the distribution at generation n?

(b) What is the distribution of X_n as $n \to \infty$? What if at generation 0, the number of individuals is an arbitrary probability distribution over the non-negative integers; does the distribution still converge? (Justify the convergence rigorously.)

3. Balls and Bins: Poisson Convergence

Consider throwing m balls into n bins uniformly at random. In this question, we will show that the number of empty bins converges to a Poisson limit, under the condition that $n\exp(-m/n) \to \lambda \in (0,\infty)$.

(a) Let $p_k(m,n)$ denote the probability that exactly k bins are empty after throwing m balls into n bins uniformly at random. Show that

$$p_0(m,n) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left(1 - \frac{j}{n}\right)^m.$$

(*Hint*: Use the Inclusion-Exclusion Principle.)

(b) Show that

$$p_k(m,n) = \binom{n}{k} \left(1 - \frac{k}{n}\right)^m p_0(m, n-k). \tag{1}$$

(c) Show that

$$\binom{n}{k} \left(1 - \frac{k}{n}\right)^m \le \frac{\lambda^k}{k!} \tag{2}$$

as $m, n \to \infty$ (such that $n\exp(-m/n) \to \lambda$).

(d) Show that

$$\binom{n}{k} \left(1 - \frac{k}{n}\right)^m \ge \frac{\lambda^k}{k!} \tag{3}$$

as $m, n \to \infty$ (such that $n\exp(-m/n) \to \lambda$). This is the hard part of the proof. To help you out, we have outlined a plan of attack:

i. Show that

$$\binom{n}{k} \left(1 - \frac{k}{n}\right)^m \ge \left(1 - \frac{k}{n}\right)^{k+m} \frac{n^k}{k!}.$$

ii. Show that

$$\ln\left\{n^k \left(1 - \frac{k}{n}\right)^m\right\} \to k \ln \lambda$$

as $m, n \to \infty$ (such that $n\exp(-m/n) \to \lambda$). You may use the inequality $\ln(1-x) \ge -x - x^2$ for $0 \le x \le 1/2$.

iii. Show that

$$\left(1 - \frac{k}{n}\right)^k \to 1$$

as $m, n \to \infty$ (such that $n\exp(-m/n) \to \lambda$). Conclude that (3) holds.

(e) Now, show that

$$p_0(m,n) \to \exp(-\lambda).$$

(Try using the results you have already proven.) Conclude that

$$p_k(m,n) \to \frac{\lambda^k \exp(-\lambda)}{k!}.$$

4. **Poisson Practice**

Let $(N(t), t \geq 0)$ be a Poisson process with rate λ. Let T_k denote the time of k-th arrival, for $k \in \mathbb{N}$, and given $0 \leq s < t$, we write $N(s,t) = N(t) - N(s)$. Compute:

(a) $\mathbb{P}(N(1) + N(2,4) + N(3,5) = 0)$.

(b) $\mathbb{E}(N(1,3) \mid N(1,2) = 3)$.

(c) $\mathbb{E}(T_2 \mid N(2) = 1)$.

5. **Basketball II**

Team A and Team B are playing an untimed basketball game in which the two teams score points according to independent Poisson processes. Team A scores points according to a Poisson process with rate λ_A and Team B scores points according to a Poisson process with rate λ_B. The game is over when one of the teams has scored k more points than the other team.

(a) Suppose $\lambda_A = \lambda_B$, and Team A has a head start of $m < k$ points. Find the probability that Team A wins.

(b) Keeping the assumptions from part (b), find the expected time $\mathbb{E}[T]$ it will take for the game to end.