EECS 126: Probability & Random Processes
Fall 2021

Multiplexing

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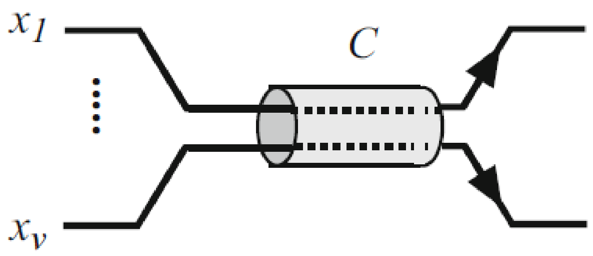
Multiplexing

- Canonical Example: Sharing of a link by multiple connections.

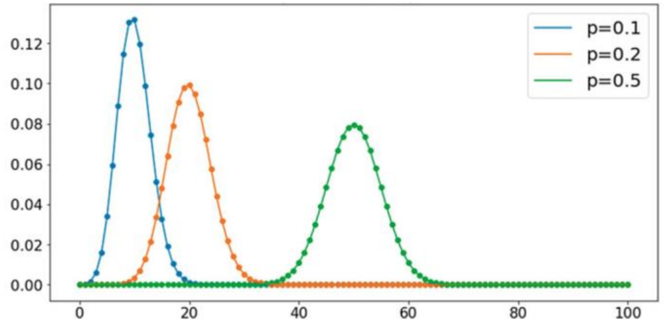


  - Each connection will get the rate of C/v, where the RV v represents # of active connections.
- In general, multiplexing addresses sharing of a common resource.
- We'll consider different statistical aspects of multiplexing.

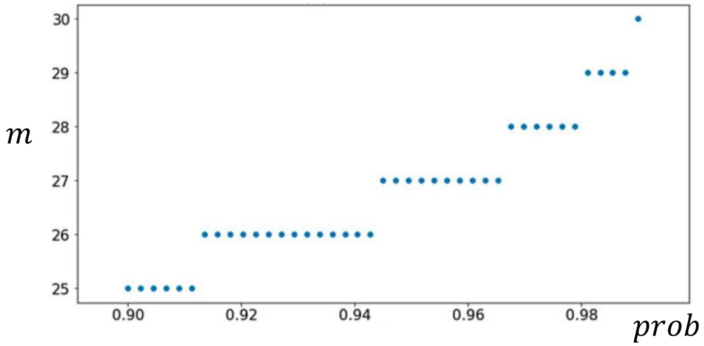
Binomial Distribution

- Consider the link sharing example.
- $v \equiv_D \; B(N, p)$ can be used to model the number of active users.

Probability Mass Function $B(100, p)$



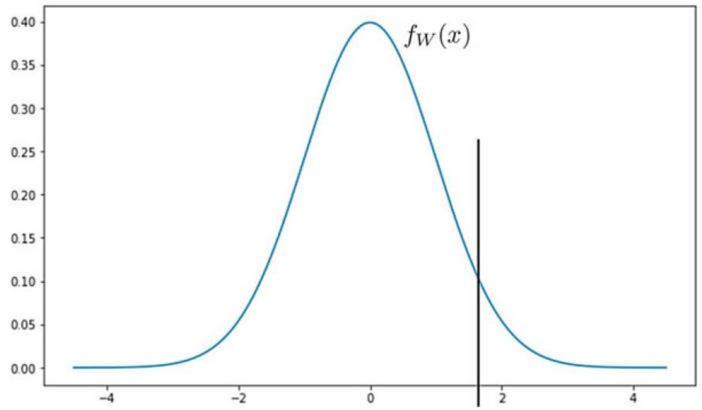
Percent Point Function (CDF Inverse) of $B(100, 0.2)$



- Find smallest m s.t. $P(v > m) \leq 0.05$, or equivalently $P(v \leq m) > 0.95$
  - This would imply each active user will get at least a rate of C/m with probability 95% or higher.
  - Observe $P(v \leq 27) = 0.966 > 0.95$ and $P(v \leq 26) = 0.944 < 0.95$.

Gaussian Random Variable

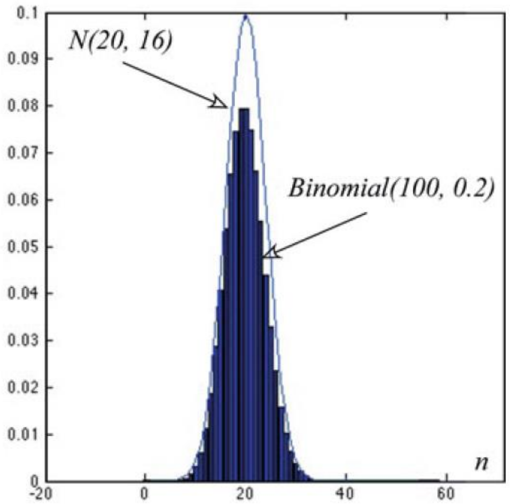
- $X =_D \mathcal{N}(\mu, \sigma^2)$
  - PDF $f_X(\mathrm{x}) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right), -\infty < x < \infty.$
  - Note $X = \mu + \sigma W$, where $W =_D \mathcal{N}(0, 1)$ is a standard normal/Gaussian RV.



  - Useful facts: $P(W > 1.65) \approx 0.05, P(W > 1.96) \approx 0.025, P(W > 2.32) \approx 0.01.$
  - Above facts hold for general Gaussian RV larger than the mean by a scalar multiple of the standard deviation (e.g., $P(X > \mu + 1.65\sigma) = 0.05$, where $X =_D \mathcal{N}(\mu, \sigma^2)$).

Central Limit Theorem (CLT)

- Convergence in Distribution: Let $\{X(n), n \geq 1\}$ and X be random variables. We say $X(n)$ converges in distribution to X, and write $X(n) \Rightarrow$ X, if $P(X(n) \leq x) \rightarrow P(X \leq x)\ \forall x\ s.t. P(X = x) = 0$.
  - $X(n) \rightarrow X\ a.s.$ implies $X(n) \rightarrow X$ in probability implies $X(n) \Rightarrow X$. (See proofs here.)
- CLT Theorem: Let $\{X(n), n \geq 1\}$ be IID RVs with mean $E\big(X(n)\big) = \mu$, and $var\big(X(n)\big) = \sigma^2$. Then, as $n \rightarrow \infty$, $\frac{X(1)+X(2)+\cdots+X(n)-n\mu}{\sigma\sqrt{n}} \Rightarrow \mathcal{N}(0,1)$.
- Binomial and Gaussian:
  - BY CLT, $B(N,p) \approx \mathcal{N}\big(Np, Np(1-p)\big)$.



  - $P\big(B(N,p) > Np + 1.65\sqrt{Np(1-p)}\big) \approx 0.05$. Hence $m = Np + 1.65\sqrt{Np(1-p)} \approx 27$ for $B(100, 0.2)$.

Confidence Intervals

- Let $Y(N) \equiv_D B(N,p)/N$, and define $A_1 = \{Y(N) \geq p + 1.65\sqrt{p(1-p)/N}\}$ and $A_2 = \{Y(N) \leq p - 1.65\sqrt{p(1-p)/N}\}$.
- Due to CLT, $P(A_1 \cup A_2) \approx 0.1$, or $P(A_1^C \cap A_2^C) \approx 0.9$.
- Since $p(1-p) < 1/4$, we have $P\left(Y(N) - 0.83\frac{1}{\sqrt{N}} \leq \mathrm{p} \leq Y(N) + 0.83\frac{1}{\sqrt{N}}\right) \geq 0.9$.
  - 90% Confidence Interval.
  - Replacing 1.65 by 2 (i.e., 0.83 by 1), gives 95% Confidence Interval.
- Increasing N, shrinks the interval.
  - Determine N based on the desired estimation margin.
- Consider any IID RVs $\{X(n), n \geq 1\}$ with mean μ without knowledge of a bound on the variance.
  - Let the sample mean $\mu_n = \frac{X(1)+X(2)+\cdots+X(n)}{n}$.
  - Let σ_n be the sample standard deviation, where $\sigma_n^2 = \frac{\sum_{m=1}^{n}(X(m)-\mu_n)^2}{n-1}$.
  - 90% Confidence Interval for μ: $\left[\mu_n - 1.65\frac{\sigma_n}{\sqrt{n}}, \mu_n + 1.65\frac{\sigma_n}{\sqrt{n}}\right]$.
  - 95% Confidence Interval for μ: $\left[\mu_n - 2\frac{\sigma_n}{\sqrt{n}}, \mu_n + 2\frac{\sigma_n}{\sqrt{n}}\right]$.

Characteristic Functions

- Definition: The characteristic function of a random variable X is defined as $\phi_X(u) = \mathrm{E}\left(e^{iuX}\right), u \in \mathcal{R}$, where $i = \sqrt{-1}$.
  - It's similar to the Moment Generating Function (MGF) $M_X(t) \coloneqq E(e^{tX}), t \in \mathcal{R}$.
  - Let $X \equiv_D \mathcal{N}(0, 1)$. Then, $\phi_X(u) = e^{-\frac{u^2}{2}}$.
  - A characteristic function determines the associated PDF/CDF uniquely.
- Moments of $\mathcal{N}(0, 1)$.

Proof of Central Limit Theorem (Sketch)

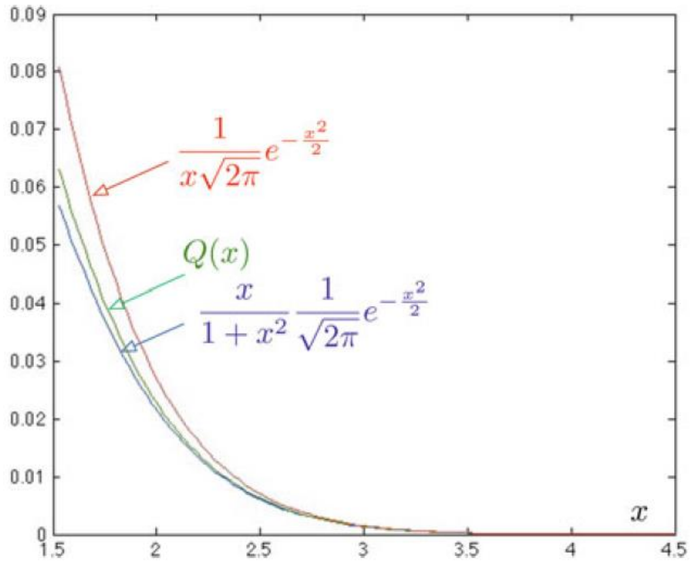
- Theorem: Let $\{X(n), n \geq 1\}$ be IID RVs with mean $E\big(X(n)\big) = \mu$, and $var\big(X(n)\big) = \sigma^2$. Then, as $n \to \infty$, $\frac{X(1)+X(2)+\cdots+X(n)-n\mu}{\sigma\sqrt{n}} \Rightarrow \mathcal{N}(0,1)$.

Two Applications of Characteristic Functions

- Poisson as a limit of Binomial: $B(n, \lambda/n) \Rightarrow P(\lambda)$.
- Exponential as a limit of Geometric: $G(\lambda/n)/n \Rightarrow Exp(\lambda)$.

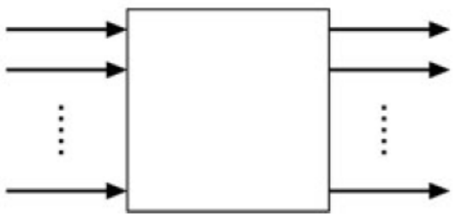
Miscellaneous Results

- Let X, Y be IID $\mathcal{N}(0,1)$ RVs. Then, $Z \coloneqq X^2 + Y^2 \equiv_D Exp(1/2)$.
- Let $Q(x) \coloneqq P(X > x)$, $X \equiv_D \mathcal{N}(0,1)$. Then, $\frac{x}{1+x^2} f_X(x) \leq Q(x) \leq \frac{1}{x} f_X(x), \forall\, x > 0$, where $f_X(x)$ is the PDF of X.



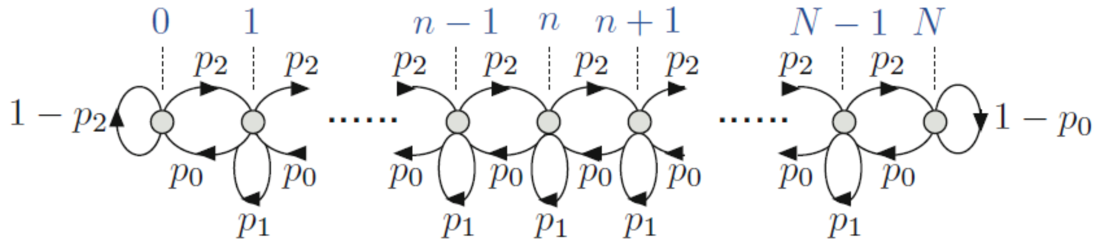
Buffers at a switch

- Multiplexing of traffic at an output port of a switch can cause buffering.



  - DTMC model for buffering at an output port.
  - At each instant n+ (right after time n)* a packet arrives with probability $\lambda \in [0, 1]$ independently of the past.
  - Time to transmit a packet is geometrically distributed with parameter $\mu \in (0, 1]$, all transmission times are independent.
    - A packet in service completes transmission at n- (just before time n)* with probability μ.
  - Let $\{X_n, n \geq 0\}$ be the number of packets in the output buffer at time n.

$$p_0 = \mu(1-\lambda), p_2 = \lambda(1-\mu), p_1 = 1 - p_0 - p_2$$



*This clarification of arrival/departure at n+/n- is not in the textbook.

Buffers at a switch (Cont'd)

- DTMC Analysis:
  - Let N be the buffer capacity in terms of number of packets.
  - Balance equations yield the invariant distribution π with $\pi(i) = \pi(0)\rho^i, \mathrm{i} = 0, 1, \ldots, \mathrm{N}$, where $\rho = p_2/p_0$, and $\pi(0) = \frac{1-\rho}{1-\rho^{N+1}}$.
  - Average # of packets in the buffer under the invariant distribution $\approx \frac{\rho}{1-\rho}$ (assuming $\lambda < \mu$ and $N \gg 1$).
  - Average delay in the buffer (from arrival until service completion) $\approx \frac{1-\mu}{\mu-\lambda}$.
- Little's Law: Let L = average # in the system, λ = average arrival rate and W = average time in the system, then $L = \lambda W$.