EECS 126: Probability & Random Processes
Fall 2021

Networks

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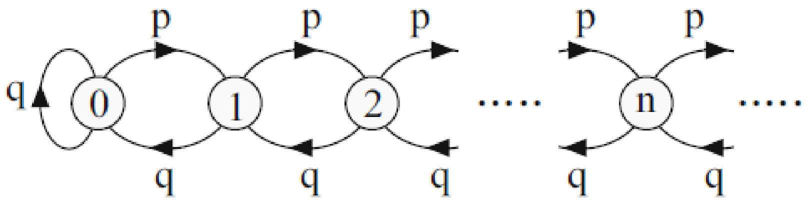
Topics of Interest on Networks

- Infinite Discrete Time Markov Chains (Section 15.3)
- Poisson Process (Section 15.4)
- Continuous Time Markov Chains (Section 6.2)
- Queues (Sections 5.6, 5.7, 5.10, 6.3)
- (Optional Reading) Social Networks: Spreading Rumors (Section 5.1), Cascades (Section 5.2)

Infinite Discrete Time Markov Chains (DTMCs)

- $\{X(n), n \geq 0\}$ is a Markov Chain over an infinite State Space $\mathcal{X} = \{0, 1, 2, \ldots\}$.
  - Initial distribution $\pi(i), i \in \mathcal{X}$ s. t. $\pi(i) \geq 0, \sum_i \pi(i) = 1$.
  - State Transition Probability Matrix P of non-negative numbers s. t. $\sum_j P(i,j) = 1, \forall\, i$.
- Irreducible and aperiodic DTMCs are defined the same way as for the finite DTMCs.
- Invariant distribution π satisfies the balance equations $\pi = \pi P$.
- A state is **transient** if one starts from this state, it's visited only finitely often. A state is **recurrent** if it's not transient.
- A recurrent state is **positive recurrent** if the average time between successive visits is finite, other wise it's **null recurrent**.
- Theorem: For an irreducible DTMC, states are either all transient, all positive recurrent or all null recurrent.
- Example: Random Walk reflected at 0.

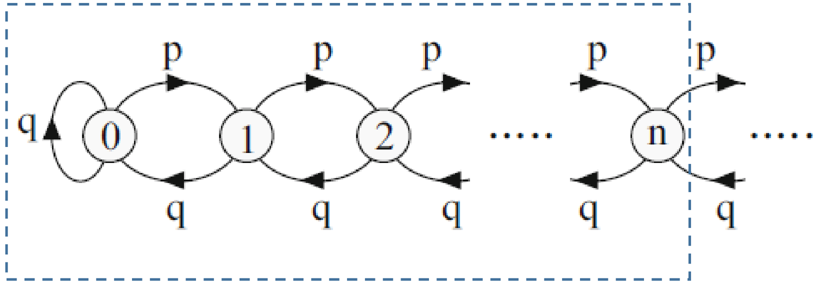


  - Transient if $p > 1/2$, null recurrent if $p = 1/2$, and positive recurrent if $p < 1/2$.

Big Theorem for Infinite DTMC

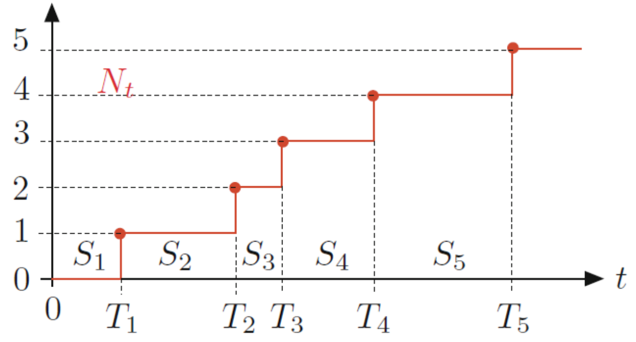
- Theorem: Consider an irreducible DTMC over an infinite state space with an invariant distribution π. Then, for each i, $\pi(i) = 1/E[T_i | X(0) = i]$, where T_i is the first time > 0 to reach state i.
- Big Theorem: Consider an irreducible DTMC over an infinite state space. Then,
  a. If positive recurrent, there is a unique invariant distribution π.
  b. If positive recurrent, long-term fraction of time $(X(n) = i) := \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} 1\{X(n) = i\} = \pi(i)$.
  c. If positive recurrent and aperiodic, $\pi_n \to \pi$.
  d. If not positive recurrent, it does not have an invariant distribution, and fraction of time spent in any state goes to 0.

Random Walk reflected @ 0.



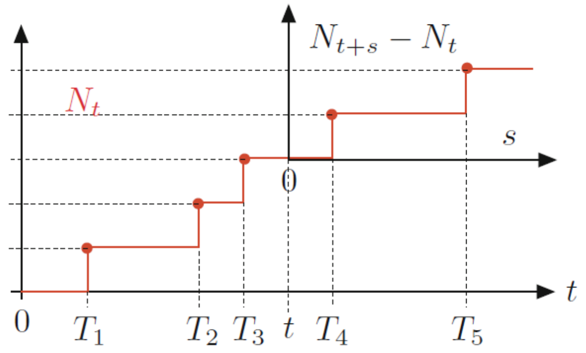
Poisson Process

- Definition: Let $\lambda > 0$, and $\{S_1, S_2, \dots\}$ be IID $Exp(\lambda)$ RVs. Let also $T_n = S_1 + \cdots + S_n$ for $n \geq 1$. Define $N_t = 0$ if $t < T_1$, otherwise $N_t = \max\{\text{n} \geq 1 \mid T_n \leq t\}$, $\text{t} \geq 0$. Then, $N \coloneqq \{N_t, t \geq 0\}$ is a Poisson process with rate λ.



- Theorem (Poisson process is Memoryless): Let $N \coloneqq \{N_t, t \geq 0\}$ be a Poisson process with rate λ. Given $\{N_s, s \leq t\}$, $\{N_{s+t} - N_t, s \geq 0\}$ is Poisson process with rate λ.

- Corollary: The process has stationary and independent increments.

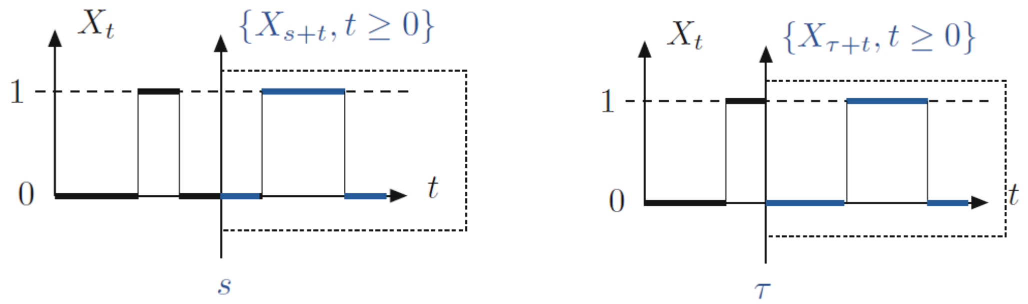


Number of Jumps

- Theorem: Let $N \coloneqq \{N_t, t \geq 0\}$ be a Poisson process with rate λ. Then $N_t \equiv_D P(\lambda t)$.
- Corollary: Given $N_t = n$ with $n \geq 1$, "unordered" jump epochs are IID uniform over $(0, \mathrm{t})$.

Continuous Time Markov Chain (CTMC)

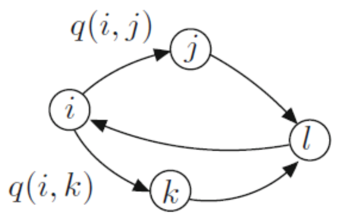
- Let $\mathcal{X}$ be a finite or countable state space, and define a rate matrix $Q = \{q(i,j), i,j \in \mathcal{X}\}$ s. t. $q(i,j) \geq 0, \forall\, i \neq j$ and $\sum_j q(i,j) = 0, \forall\, i$.
- Definition: A CTMC with initial distribution π_0 and rate matrix Q is a process $\{X_t, t \geq 0\}$ s. t. $P(X_0 = i) = \pi_0(i)$, and $P(X_{t+\epsilon} = j \mid X_t = i, X_u, u < t) = 1\{\mathrm{i} = \mathrm{j}\} + \epsilon q(i,j) + o(\epsilon)$.
- Stopping Time and Strong Markov Property

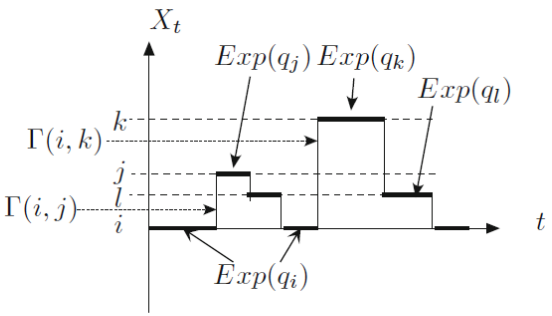


Starts afresh after s

Starts afresh after stopping time τ

- Construction:
  - If $X_t = i$, choose random τ that's exponentially distributed with rate $q_i = -q(i,i) = \sum_{j \neq i} q(i,j)$, and time $t + \tau$, jump to state $j \neq i$ with probability $\Gamma(i,j) := q(i,j)/q_i$.





CTMC Transient and Invariant Distributions

- Let π_t be the distribution of X_t.
  - Note $\pi_{t+\epsilon}(i) \approx \sum_{j\neq i} \pi_t(j)q(j,i)\epsilon + \pi_t(\mathrm{i})(1-q_i\epsilon)$, or $\pi_{t+\epsilon} \approx \pi_t(I+Q\epsilon)$.
  - Hence, $\frac{d}{dt}\pi_t = \pi_t Q$ (Kolmogorov Forward Equation) $\Rightarrow \pi_t = \pi_0 e^{Qt}$, where $e^{Qt} = I + Qt + \frac{1}{2!}Q^2t^2 + \frac{1}{3!}Q^3t^3 + \cdots$ (Taylor Series)
  - This gives $\pi_t = \pi \ \forall\, t \geq 0 \Leftrightarrow \pi_0 = \pi, \pi Q = 0$.
  - Invariant distribution π satisfies $\pi Q = 0$ (flow-in = flow-out) and $\sum_i \pi(i) = 1$.
- CTMC examples: Two-state example, Poisson Process

Big Theorem for CTMC

- **Theorem:** For an irreducible CTMC, states are either all transient, all positive recurrent or all null recurrent.
- Big Theorem: Consider an irreducible CTMC over a finite or countable state space. Then,

a. If positive recurrent, there is a unique invariant distribution π.

b. If positive recurrent, long-term fraction of time $(X_t = i) \coloneqq \lim_{t\to\infty} \frac{1}{t} \int 1\{X_t = i\}\, dt = \pi(i)$.

c. If positive recurrent, $\pi_t \to \pi$.

d. If not positive recurrent, it does not have an invariant distribution, and fraction of time spent in any state goes to 0.

Three DTMCs associated with a CTMC

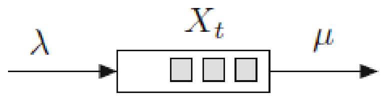
- Discrete Time Approximation:
  - Since $\pi_{t+\epsilon} \approx \pi_t(I + Q\epsilon)$, $\{X_{n\epsilon}, n = 0, 1, \dots\}$ is a DTMC with transition probability matrix $P = I + Q\epsilon$.
  - Same invariant distribution as that of the CTMC.
- Embedded DTMC (or Jump DTMC):
  - Transition Probability Matrix: $\Gamma(i,j) = \begin{cases} q(i,j)/q_i, if\ i \neq j \\ 0, if\ i = j \end{cases}$
  - Let υ be it's invariant distribution, i.e., $\upsilon = \upsilon\Gamma$.
  - Relationships between υ and π (the invariant distribution of X_t).
    - $\pi(i) = \frac{\upsilon(i)/q_i}{\sum_k \upsilon(k)/q_k}$
    - $\upsilon(i) = \frac{\pi(i)q_i}{\sum_k \pi(k)q_k}$
- Uniformized DTMC:
  - Fix $\lambda \geq q_i\ \forall\ i$, and define $P(i,j) = \begin{cases} q(i,j)/\lambda, if\ i \neq j \\ 1 - q_i/\lambda, if\ i = j \end{cases}$
    - I.e., $P = I + \frac{1}{\lambda}Q$.
  - Observe $\pi P = \pi \Leftrightarrow \pi Q = 0$.
  - Applications:
    - Invariant distribution of X_t can be found by computing P^k as $k \to \infty$.
    - Transient distribution of X_t: Define CTMC Y_t with inter-jump times being IID $Exp(\lambda)$ and jump probabilities given by P. Then, $\pi_t = \sum_{n=0}^{\infty} \pi_0 P^n \frac{(\lambda t)^n e^{-\lambda t}}{n!}$.

Reversibility

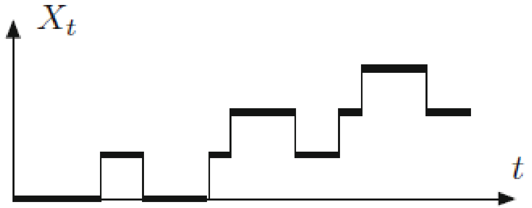
- **Theorem:** Assume a CTMC X_t has the invariant distribution π. Then, X_t reversed in time is a CTMC with the same invariant distribution, and it's rate matrix $\tilde{Q}$ is given by $\tilde{q}(i,j) = \frac{\pi(j)q(j,i)}{\pi(i)}$.
  - Note: CTMCs in the forward and reversed directions have the same invariant distributions
- Suggests the following recipe:
  - Guess the invariant distribution π for the CTMC under consideration.
  - Guess the CTMC in reversed time and find it's rate matrix $\tilde{Q}$.
  - Show the equation in the theorem above is satisfied.
  - Proves our guesses for the invariant distribution and the CTMC in reversed time are correct.
- If a CTMC satisfies the detailed balance equations, it's reversible (i.e., the rate matrices in the forward and reversed directions are the same).

M/M/1 Queue

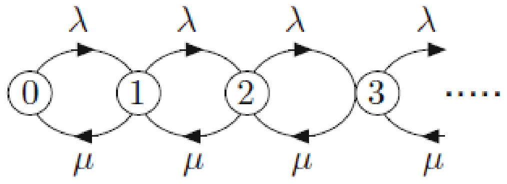
- Packets/customers arrive at a single server queue according to a Poisson process with rate $\lambda > 0$, and their service times are IID $Exp(\mu)$.



A realization:



State Transition Diagram:



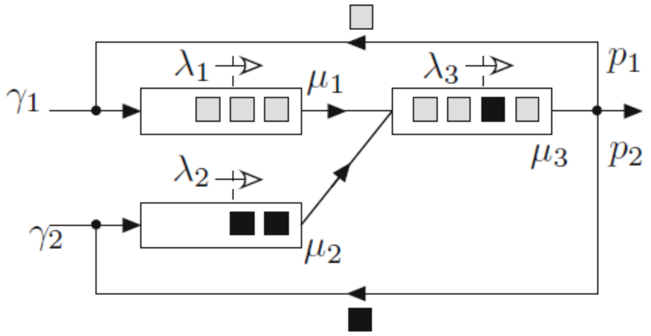
- CTMC is transient, null recurrent or positive recurrent if $\lambda > \mu, \lambda = \mu$ or $\lambda < \mu$, respectively.
- Assume $\lambda < \mu$. Then,
  - Invariant Distribution: $\pi(n) = (1 - \rho)\rho^n, n \geq 0$, where $\rho = \lambda/\mu$.
  - Average # in the queue under the invariant distribution $= \frac{\rho}{1-\rho}$.
  - Average delay in the system $\frac{1}{\mu-\lambda}$.

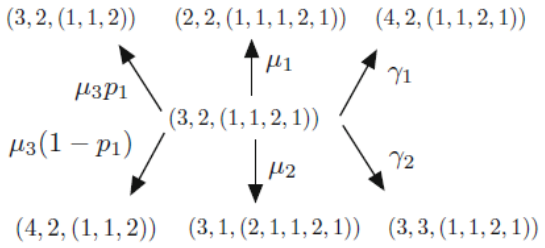
Network of Queues

- Consider a network of queues with $N \geq 1$ queues and $C \geq 1$ classes:
  - Let $X(t) = \{X_i(t), i = 1, \dots, N\}$ where $X_i(t)$ denotes configuration of queue i at time t.
    - $X_i(t) = 321231$ indicates class for each customer in the queue i from tail to head.
  - External arrivals at queue i for class c occur according to a Poisson process with rate γ_i^c.
  - All service times at queue i are IID $Exp(\mu_i)$.
  - After service completion at queue i, a customer of class c is routed independently to queue j as class d with probability $r_{i,j}^{c,d}$.
    - Network departure probability after service at queue i is $r_{i,0}^c = 1 - \sum_{j=1}^N \sum_{d=1}^C r_{i,j}^{c,d}$.
  - Let λ_i^c be the rate into queue i for class c satisfying the flow-conservation equations $\lambda_i^c = \gamma_i^c + \sum_{j=1}^N \sum_{d=1}^C \lambda_j^d r_{j,i}^{d,c}$.
  - **Theorem:** If the network is open (i.e., each customer eventually departs), and $\lambda_i = \sum_{c=1}^C \lambda_i^c < \mu_i$, $X(t)$ is a CTMC with the unique invariant distribution π s. t.
    - $\pi(x) = \prod_{i=1}^N \pi_i(x_i)$, where x is a collection of configurations $\{x_i, i = 1, \dots, N\}$ at different queues, and
    - $\pi_i(x_i = c_1 c_2 \dots c_n) = \left(1 - \frac{\lambda_i}{\mu_i}\right) \frac{\lambda_i^{c_1} \dots \lambda_i^{c_n}}{\mu_i^n} = p_i(c_1) \dots p_i(c_n)(1-\rho_i)\rho_i^n$ with $\rho_i = \lambda_i / \mu_i$ and $p_i(c) = \lambda_i^c / \lambda_i$.
  - **Corollary:** Let L_i be the queue length at queue i. Then, $\pi(L_i = n_i, i = 1, \dots, N) = \prod_{i=1}^N (1-\rho_i)\, \rho_i^{n_i}, \rho_i = \lambda_i / \mu_i$.

Network of Queues (Example)

- Consider the following network:
  - There are three queues and two types of customers.
  - External arrivals occur according to Poisson processes.
  - All service times at queue i are IID $Exp(\mu_i)$.





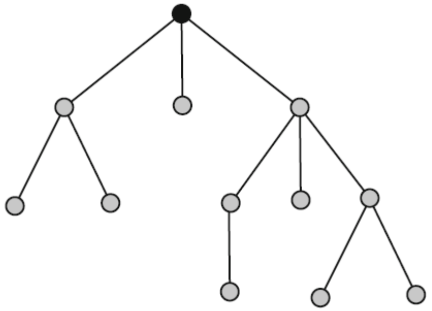
Useful Facts

- Residual Time Paradox: Suppose inter-event times are IID non-negative RVs $X_i, \mathrm{i} = 1, 2, \ldots$ with PDF $f(x)$ and CDF $F(x)$, and i^{th} moment $m_i, \mathrm{i} = 1, 2, \ldots$
  - After the process has been running for a long time, an observer arrives at an arbitrary time.
  - The inter-event time during which the observer arrives has the PDF $\frac{xf(x)}{m_1}$.
  - The residual time to the next event and age time from the last event have the PDF $\frac{1-F(x)}{m_1}$.
- Sum of independent RVs X and Y: $Z = X + Y$.
  - $f_Z(\mathrm{z})dz = \int_{-\infty}^{\infty} f_X(\mathrm{x}) f_Y(z-x)dxdz \Rightarrow f_Z(z) = \int_{-\infty}^{\infty} f_X(\mathrm{x}) f_Y(z-x)dx$
  - $f_Z(\mathrm{z}) = f_X * f_Y(\mathrm{z})$ (i.e., f_Z is convolution of f_X and f_Y).
  - Characteristic functions: $\phi_Z(u) = \phi_X(u)\phi_Y(u)$
- Sum of IID $Exp(\lambda)$ RVs: $Z = X_1 + \cdots + X_n$.
  - Z has the Erlang distribution (a special case of the Gamma distribution).
  - $Z \equiv_D \Gamma(\mathrm{n}, \lambda)$ (PDF $f_Z(z) = \frac{\lambda^n z^{n-1} e^{-\lambda z}}{(n-1)!}, \mathrm{z} \geq 0$).
- Merging of Poisson processes:
  - Let $N_1(t), \ldots, N_m(t)$ be m independent Poisson processes with rates $\lambda_1, \ldots, \lambda_m$.
  - Then, $N(t) = N_1(t) + \cdots + N_m(t)$ is a Poisson process with rate $\lambda_1 + \cdots + \lambda_m$.
- Splitting (thinning) of a Poisson process:
  - Let $N(t)$ be a Poisson process with rate λ.
  - Each arrival is included in the process $N_1(t)$ if an independent $B(p)$ coin flip results in heads, otherwise it's included in $N_2(t)$.
  - $N_1(t)$ and N_2(t) are independent Poisson processes with rates λp and $\lambda(1-p)$, respectively.

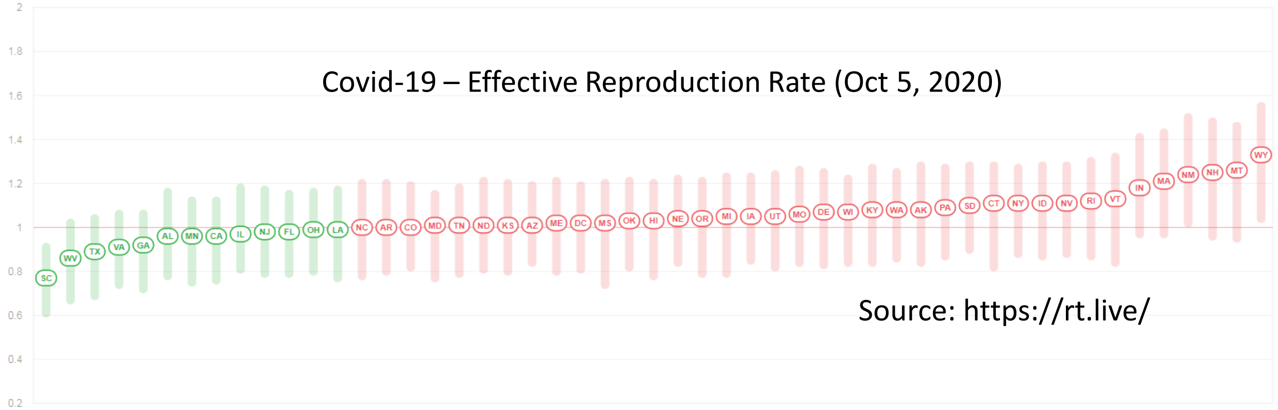
Optional: Spreading Rumors

- Number of children of different nodes are IID with mean μ.

Spreading of a Message
as a Random Tree

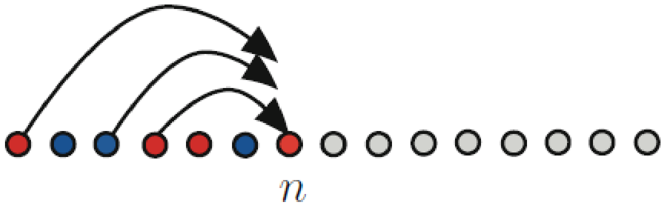


- Theorem: Let Z be the number of nodes receiving the message.
  a. If $\mu < 1$, $P(Z < \infty) = 1$, and $E(Z) < \infty$.
  b. If $\mu > 1$, $P(Z = \infty) > 0$.



Optional: Cascades

- Suppose nodes are arranged linearly, and each node needs to make a selection out of two choices.
  - Assume node 0 is red.
- Node n listens to the advice of node $n - k$ independently with probability p_k.
  - Majority dictates the choice, and flips a fair coin in case of a tie.
- How far influence cascades?



- Theorem: Suppose $p_k = p \in (0,1]\ \forall\ k \geq 1$. Then all nodes turn red with probability at least θ, where $\theta = \exp(-\frac{p}{1-p})$.