EECS 126: Probability & Random Processes
Fall 2021

Little’s Law + Jackson Networks

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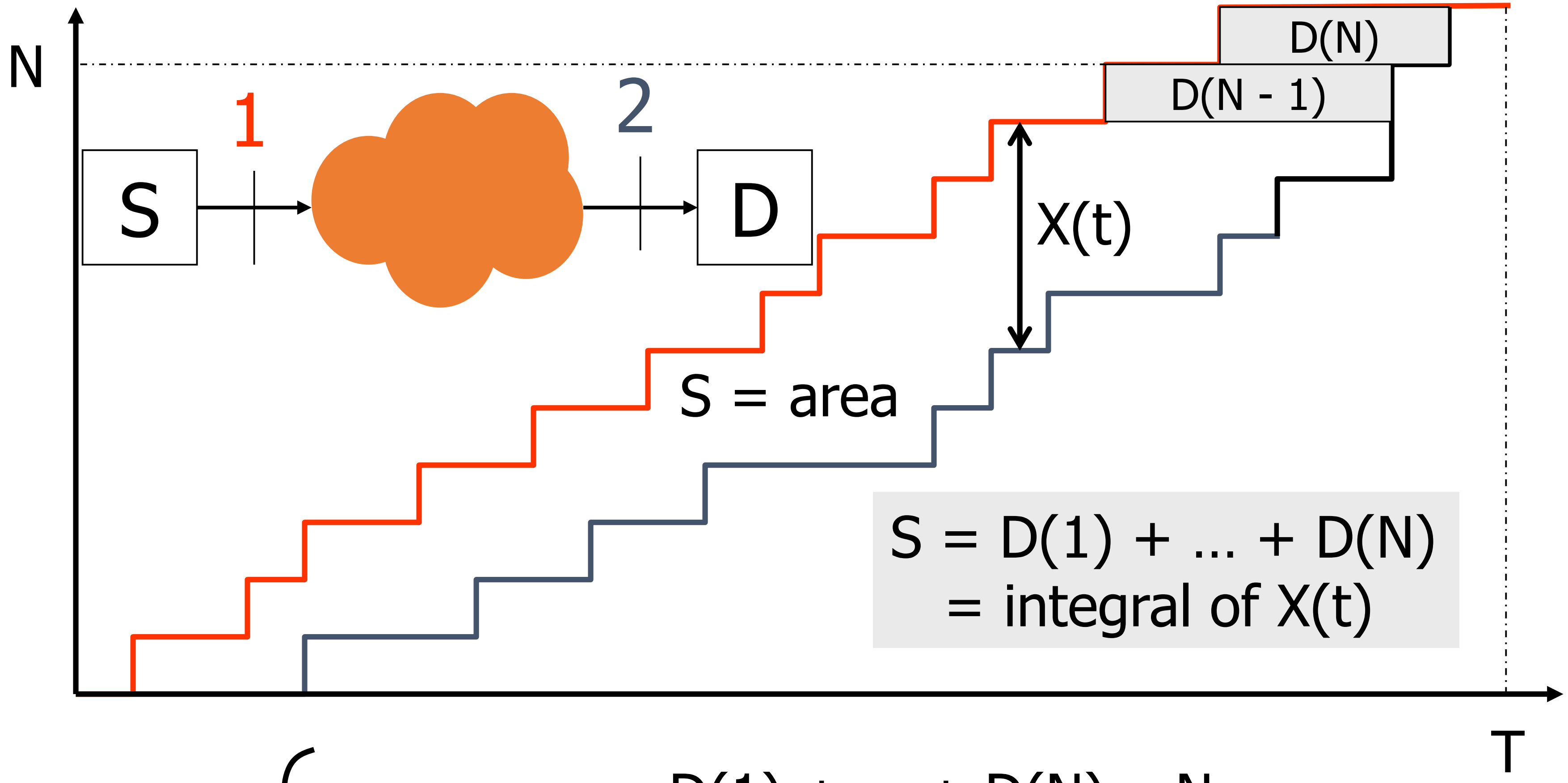
Fundamental Result: Little's Law

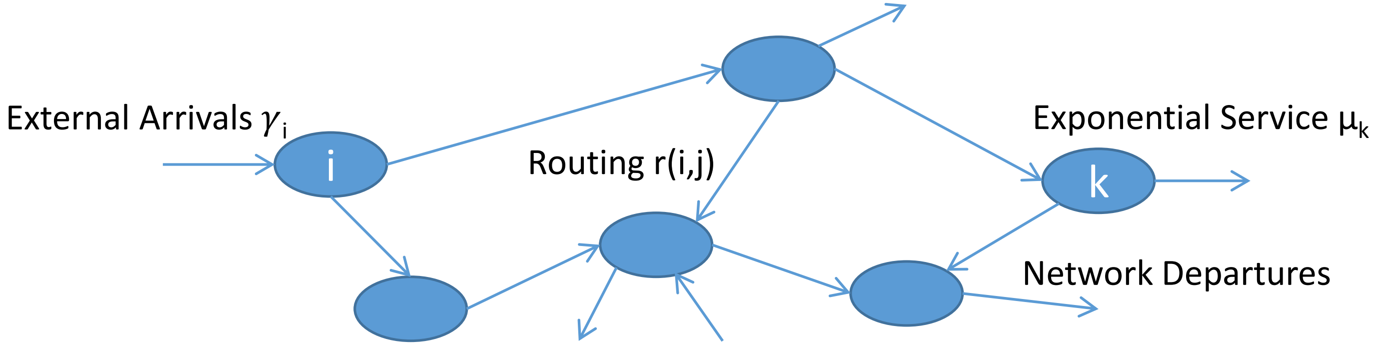


$$\frac{1}{T}\int X(t)dt = \frac{S}{T} = \frac{D(1) + \ldots + D(N)}{N} \cdot \frac{N}{T}$$

→ Average occupancy = (average delay)x(average arrival rate)

Jackson Networks

- Definition:
  - Network of J ./M/1 queues,
  - External arrivals occur according to independent Poisson processes with rate γ_i into queue i,
  - Service time at queue i is according to independent exponential distribution with rate μ_i, and
  - When a customer leaves queue i, independent of the past, he joins queue j with probability r(i,j) and leaves the network with probability $1 - \sum_{j=1}^{J} r(i,j)$.



Jackson Networks (2)

- We assume that routing is such that every customer eventually leaves the network.
- Due to flow conservation, total arrival rate into queue i is given by

$$\lambda_i = \gamma_i + \sum_{j=1}^{J} \lambda_j r(j, i), \text{ for i = 1, 2, ... J.}$$

- For t ≥ 0, we define $X_t = (X_{1,t}, \dots, X_{J,t})$, where $X_{i,t}$ denotes total # of customers at node i.
  - X_t is a multidimensional CTMC.

Jackson Networks (3)

- Theorem: Assume that the solution $(\lambda_1, \ldots, \lambda_J)$ is such that $\lambda_i < \mu_i$ for i = 1, …, J. Then, the CTMC X_t admits the following invariant distribution:

$$\pi\left(x_1, \ldots, x_J\right) = \pi_1(x_1). \ldots . \pi_J(x_J), \text{ where}$$

For each j, $\pi_j(n) = (1 - \rho_j)\rho_j{}^n$, for n ≥ 0 and $\rho_j = \frac{\lambda_j}{\mu_j}$.

- I.e., each queue behaves like an M/M/1 queue with appropriate utilization.
- Referred to as Product Form Networks