1. Among Us

In one game of Among Us, there are 9 players, consisting of 4 imposters and 5 crewmates. There is also a deck of 17 cards, containing 11 “sabotage” cards and 6 “task” cards. Imposters want to play sabotage cards, and crewmates want to play task cards. Here’s how the play proceeds.

• A captain and a first mate are chosen uniformly at random from the 9 players.
• The captain draws 3 cards from the deck and gives 2 to the first mate, discarding the third.
• The first mate chooses one to play.

Now suppose that you are the first mate, but the captain gave you 2 sabotage cards. Being a crewmate, you wonder, did the captain just happen to have 3 sabotage cards, or was the captain an imposter who secretly discarded a task card? In this scenario, find the probability that the captain is an imposter. Assume that imposter captains always try to discard task cards, and crewmate captains always try to discard sabotage cards.

Solution: Let \( A \) be the given event that you are handed 2 sabotage cards, and let \( B \) be the event that the captain is an imposter (sussy baka). By Bayes’ rule,

\[
P(B \mid A) = \frac{\Pr(A \mid B) \cdot \Pr(B)}{\Pr(A \mid B) \cdot \Pr(B) + \Pr(A \mid B^c) \cdot \Pr(B^c)}.\]

You are a crewmate, so \( \Pr(B) = \frac{4}{9} = \Pr(B^c) \). We note that \( \Pr(A \mid B) \) is the sum of probabilities of two cases:

• The captain drew 3 sabotage cards.
• The captain drew 2 sabotage cards and 1 task card, but discarded the task card.

On the other hand, \( \Pr(A \mid B^c) \) is the probability of the first case only. We find that

\[
\Pr(A \mid B^c) = \frac{\binom{11}{3}}{\binom{17}{3}} = \frac{11 \cdot 10 \cdot 9}{17 \cdot 16 \cdot 15} = \frac{33}{136},
\]

\[
\Pr(A \mid B) = \frac{33}{136} + \frac{\binom{11}{2} \cdot \binom{6}{1}}{\binom{17}{3}} = \frac{33}{136} + \frac{\frac{11 \cdot 10}{2} \cdot 6}{17 \cdot 16 \cdot 15} = \frac{99}{136}.
\]

After simplifying, we get that

\[
\Pr(B \mid A) = \frac{99}{99 + 33} = \frac{3}{4}.
\]

In other words, it is quite likely that the captain is an imposter among us :flushed:
2. Random Walk on a Circle

Suppose we have \(n\) points labeled \(\{1, 2, \ldots, n\}\) around a circle. An ant starts at point 1, and at each second has an equal probability of moving clockwise or counterclockwise to an adjacent point. For each point \(k \in \{2, \ldots, n\}\), find the probability that the first time the ant lands at \(k\), it has visited all other points already.

*Hint:* Try simulating this problem. What do you observe? To justify your observations, consider \(\tau_k\), the first time the ant lands on a neighbor of \(k\), and the events

\[
A_k := \{\text{the ant lands on a neighbor of } k \text{ at some time}\} = \{\tau_k < \infty\}
\]

\[
B_k := \{\text{starting at time } \tau_k, \text{ the ant visits all other points before first landing on } k\}.
\]

**Solution:** Fix \(k \in \{2, \ldots, n\}\), and let \(C_k\) be the event that the ant starts from 1 and visits all other points before landing in \(k\) for the first time. We claim that \(\mathbb{P}(C_k) = \frac{1}{n-1}\). While it is clear that some symmetry is involved, it is not immediately clear that \(\mathbb{P}(C_k)\) is the same for all \(k\) by symmetry, as some points are farther from the starting point 1 than others.

So we will need to define some events in order to formally find our answer. Let \(\tau_k\) be the first time in which the ant lands on a neighbor of \(k\), and consider the events

\[
A_k := \{\text{the ant lands on a neighbor of } k \text{ at some time}\} = \{\tau_k < \infty\}
\]

\[
B_k := \{\text{starting at time } \tau_k, \text{ the ant visits all other points before first landing on } k\}.
\]

These are simply the two conditions for \(C_k\): to visit \(k\), the ant must first land on a neighbor of \(k\), and from there must have also visited the other neighbor of \(k\) before landing on \(k\) itself. Thus

\[
\mathbb{P}(C_k) = \mathbb{P}(A_k \cap B_k) = \mathbb{P}(B_k | A_k) \cdot \mathbb{P}(A_k) = \mathbb{P}(B_k | A_k).
\]

We leave the reader to check that \(\mathbb{P}(A_k) = 1\). Now, we notice that by rotational symmetry, \(\mathbb{P}(B_k | A_k) = \mathbb{P}(C_2)\), the probability that the ant starts at 1 and visits all other points before first landing on 2. Since \(\mathbb{P}(C_2) = \mathbb{P}(C_k)\) for all \(k \in \{2, \ldots, n\}\), and

\[
\mathbb{P}(C_2) + \cdots + \mathbb{P}(C_n) = 1,
\]

we have \(\mathbb{P}(C_k) = \frac{1}{n-1}\) as claimed.
3. Colored Sphere

Consider a sphere that has $\frac{1}{10}$ of its surface colored blue, and the rest colored red. Show that no matter how the colors are distributed, it is possible to inscribe a cube in the sphere with all of its vertices red.

**Hint:** Carefully define some relevant events. Observe that if $\mathbb{P}(A) > 0$, then the event $A$ must be nonempty.

**Solution:** Pick an inscribed cube uniformly at random, enumerate its vertices $1, \ldots, 8$, and let $B_i$ be the event that vertex $i$ is blue. Note that

$$
\mathbb{P}\left(\bigcup_{i=1}^{8} B_i\right) \leq \sum_{i=1}^{8} \mathbb{P}(B_i) = \sum_{i=1}^{8} \frac{1}{10} = \frac{8}{10} < 1.
$$

In other words, the probability of at least one vertex being blue is strictly less than 1, so the probability that no vertex is blue (every vertex is red) is strictly greater than 0. Because a randomly sampled inscribed cube has nonzero probability of having all vertices red, there must exist at least one inscribed cube with all vertices red.

**Note:** this is an example of a powerful mathematical technique known as the **probabilistic method**, primarily used in combinatorics to show the existence of some kind of object.