1. Running Sum of a Markov Chain

Let \((X_n)_{n \in \mathbb{N}}\) be a Markov chain with two states, \(-1\) and \(1\), and transition probabilities \(P(-1,1) = P(1,-1) = a\) for \(a \in (0,1)\). Define

\[ Y_n = X_0 + X_1 + \cdots + X_n. \]

For what values of \(a\) is \((Y_n)_{n \in \mathbb{N}}\) a Markov chain?

**Solution:** \((Y_n)_{n \in \mathbb{N}}\) is not a Markov chain if \(a \neq \frac{1}{2}\). Consider the following probability:

\[ P(Y_4 = 3 \mid Y_2 = 1, Y_3 = 2) = P(X_4 = 1 \mid X_3 = 1) = 1 - a. \]

On the other hand,

\[ P(Y_4 = 3 \mid Y_2 = 3, Y_3 = 2) = P(X_4 = 1 \mid X_3 = -1) = a, \]

so the distribution of \(Y_4\) given the past is not dependent solely on \(Y_3\), which shows that the Markov property does not hold.

However, \((Y_n)_{n \in \mathbb{N}}\) is a Markov chain when \(a = \frac{1}{2}\), as \((X_n)_{n \in \mathbb{N}}\) are then i.i.d. with \(P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2}\). So, for any positive integer \(n\),

\[ Y_n = \begin{cases} 
Y_{n-1} + 1 & \text{with probability } \frac{1}{2} \\
Y_{n-1} - 1 & \text{with probability } \frac{1}{2}.
\end{cases} \]

In other words, given \(Y_{n-1}\), \(Y_n\) is conditionally independent of all previous states \(Y_0, \ldots, Y_{n-2}\), so \((Y_n)_{n \in \mathbb{N}}\) is a Markov chain.
2. Doubly Stochastic Matrix

A matrix is called **doubly stochastic** if all of its entries are nonnegative, and each row and each column sums to 1. Show that any doubly stochastic matrix is a valid transition probability matrix for a Markov chain. Then, prove that the stationary distribution for a doubly stochastic irreducible matrix is uniform over the state space.

**Solution:** By definition, a doubly stochastic matrix has nonnegative entries which sum to 1 in each row, so it is a valid transition probability matrix for a Markov chain. Now, we check that \( \pi(x) := |X|^{-1} \) for every \( x \) in the state space \( X \) is a stationary distribution:

\[
\sum_{y \in X} \pi(y)P(y, x) = |X|^{-1} \sum_{y \in X} P(y, x) = |X|^{-1} = \pi(x).
\]
3. Markov Chain Practice

Consider a Markov chain with three states 0, 1, and 2. The transition probabilities are 
\[ P(0, 1) = P(0, 2) = \frac{1}{2}, \ P(1, 0) = P(1, 1) = \frac{1}{2}, \ P(2, 0) = \frac{2}{3}, \text{ and } P(2, 2) = \frac{1}{3}. \]

a. Classify the states in the chain. Is this chain periodic or aperiodic?
b. In the long run, what fraction of time does the chain spend in state 1?
c. Suppose that \( X_0 \) is chosen according to the steady-state or stationary distribution. What is \( P(X_0 = 0 \mid X_2 = 2) \)?

Solution:

a. The Markov chain has one recurrent, aperiodic class.
b. By solving \( \pi P = \pi \), we have
\[ \pi = \frac{1}{11} \begin{bmatrix} 4 & 4 & 3 \end{bmatrix}. \]
Thus \( \pi(1) = 4/11. \)
c. By the definition of conditional probability,
\[ P(X_0 = 0 \mid X_2 = 2) = \frac{P(X_0 = 0, X_2 = 2)}{P(X_2 = 2)} = \frac{P(X_0 = 0, X_1 = 2, X_2 = 2)}{P(X_2 = 2)}. \]
Note that we exploit the structure of the Markov chain in the last equality: the only possible two-step path from \( X_0 = 0 \) to \( X_2 = 2 \) is \( 0 \to 2 \to 2 \). Now, \( P(X_2 = 2) = P(X_0 = 2) \) as \( X_0 \) is chosen according to the stationary distribution \( \pi \), so
\[ \frac{P(X_0 = 0, X_1 = 2, X_2 = 2)}{P(X_2 = 2)} = \frac{\pi(0) \cdot (1/2) \cdot (1/3)}{\pi(2)} = \frac{2}{9}. \]
4. Reducible Markov Chain

Consider the following Markov chain, where $\alpha, \beta, p, q \in (0, 1)$.

![Markov Chain Diagram]

a. Find all the recurrent and transient classes.
b. Given that we start in state 2, what is the probability we reach state 0 before state 5?
c. What are all of the possible stationary distributions of this chain? *Hint:* Consider the recurrent classes.
d. Suppose we start in the initial distribution $\pi_0 := [0 \ 0 \ \gamma \ 1 - \gamma \ 0 \ 0]$ for some $\gamma \in [0, 1]$. Does the distribution of the chain converge, and if so, to what?

**Solution:**

a. The classes are $\{0, 1\}$ (recurrent), $\{4, 5\}$ (recurrent), and $\{2, 3\}$ (transient).
b. Let $T_0$ and $T_5$ denote the time it takes to reach states 0 and 5 respectively. Note that exactly one of $T_0$ and $T_5$ will be finite. We can set up hitting time equations to compute $\mathbb{P}_2(T_0 < T_5) = \mathbb{P}(T_0 < T_5 | X_0 = 2)$:

$$\mathbb{P}_2(T_0 < T_5) = \frac{1}{2} + \frac{1}{2} \mathbb{P}_3(T_0 < T_5)$$

$$\mathbb{P}_3(T_0 < T_5) = \frac{1}{2} \mathbb{P}_2(T_0 < T_5).$$

Thus $\mathbb{P}_2(T_0 < T_5) = \frac{2}{3}$.
c. First, we observe that no transient state can support nonzero probability mass at stationarity, so the stationary distribution is supported on the states $\{0, 1, 4, 5\}$. Next, if we restrict our attention to only the states $\{0, 1\}$, then we have an irreducible Markov chain with stationary distribution

$$\pi_1 := \frac{1}{\alpha + \beta} \begin{bmatrix} \beta & \alpha \end{bmatrix}.$$

Similarly, if we restrict our attention to only the states $\{4, 5\}$, we again have an irreducible Markov chain with stationary distribution

$$\pi_2 := \frac{1}{p + q} \begin{bmatrix} q & p \end{bmatrix}.$$

Any stationary distribution for the entire chain must be some convex combination of these two stationary distributions, depending on the total amount of stationary mass in each recurrent class. Explicitly, the stationary distributions are of the form

$$\pi = \begin{bmatrix} \frac{c\beta}{\alpha + \beta} & \frac{c\alpha}{\alpha + \beta} & 0 & 0 & \frac{(1-c)q}{p+q} & \frac{(1-c)p}{p+q} \end{bmatrix}$$

for $c \in [0, 1]$. 

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d. The distribution will indeed converge, even without irreducibility. Intuitively, the probability mass gradually leaves the transient states \{2, 3\}, until eventually, all of the probability mass is supported on the recurrent states. The two recurrent classes can then each be considered as an irreducible aperiodic Markov chain, so they each settle into equilibrium. Let us use the previous parts to find the limiting distribution. We start in state 2 with probability $\gamma$, and we end up in the recurrent class \{0, 1\} with further probability $\frac{2}{3}$. By symmetry, the probability that we end in \{0, 1\} starting from state 3 is $\frac{1}{3}$. Thus, the total probability mass which settles into \{0, 1\} is

$$
\frac{2\gamma}{3} + \frac{1 - \gamma}{3} = \frac{1}{3} + \frac{\gamma}{3},
$$

and the probability mass which settles in \{4, 5\} is $\frac{2}{3} - \frac{\gamma}{3}$. Therefore the chain converges to the stationary distribution found in part (c) with $c = \frac{1}{3} + \frac{\gamma}{3}$. 