1. **Running Sum of a Markov Chain**

Let \((X_n)_{n \in \mathbb{N}}\) be a Markov chain with two states, \(-1\) and \(1\), and transition probabilities \(P(-1, 1) = P(1, -1) = a\) for \(a \in (0, 1)\). Define

\[ Y_n = X_0 + X_1 + \cdots + X_n. \]

For what values of \(a\) is \((Y_n)_{n \in \mathbb{N}}\) a Markov chain?
2. Doubly Stochastic Matrix

A matrix is called **doubly stochastic** if all of its entries are nonnegative, and each row and each column sums to 1. Show that any doubly stochastic matrix is a valid transition probability matrix for a Markov chain. Then, prove that the stationary distribution for a doubly stochastic irreducible matrix is uniform over the state space.
3. Markov Chain Practice

Consider a Markov chain with three states 0, 1, and 2. The transition probabilities are $P(0, 1) = P(0, 2) = \frac{1}{2}$, $P(1, 0) = P(1, 1) = \frac{1}{2}$, $P(2, 0) = \frac{2}{3}$, and $P(2, 2) = \frac{1}{3}$.

a. Classify the states in the chain. Is this chain periodic or aperiodic?
b. In the long run, what fraction of time does the chain spend in state 1?
c. Suppose that $X_0$ is chosen according to the steady-state or stationary distribution. What is $\Pr(X_0 = 0 \mid X_2 = 2)$?
4. Reducible Markov Chain

Consider the following Markov chain, where $\alpha, \beta, p, q \in (0, 1)$.

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  0   1   2   3   4   5
\alpha \beta \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} p q
1 - \alpha 1 - \beta 1 - p 1 - q
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a. Find all the recurrent and transient classes.

b. Given that we start in state 2, what is the probability we reach state 0 before state 5?

c. What are all of the possible stationary distributions of this chain? Hint: Consider the recurrent classes.

d. Suppose we start in the initial distribution $\pi_0 := [0 \ 0 \ \gamma \ 1 - \gamma \ 0 \ 0]$ for some $\gamma \in [0, 1]$. Does the distribution of the chain converge, and if so, to what?