Continuous-Time Markov Chains: Introduction

Consider the continuous-time Markov chain defined on the state space \{1, 2, 3, 4\} which has transition rate matrix

\[
Q = \begin{bmatrix}
-3 & 1 & 1 & 1 \\
0 & -3 & 2 & 1 \\
1 & 2 & -4 & 1 \\
0 & 0 & 1 & -1
\end{bmatrix}.
\]

1. a. Find the stationary distribution \(\pi\) of this chain.

b. Find the stationary distribution \(\mu\) of the jump chain, the discrete-time Markov chain which only keeps track of the jumps of the CTMC. Formally, if \((X_t)_{t \geq 0}\) transitions at times \(T_1, T_2, \ldots\), then its jump chain is defined as \((Y_n)_{n=1}^{\infty}\), where \(Y_n := X_{T_n}\).

c. Suppose the chain starts in state 1. What is the expected amount of time until it changes state for the first time?

d. Suppose still the chain starts in state 1. What is the expected amount of time until the chain is in state 4?
2. Reversibility of CTMCs

We say that a CTMC with transition rate matrix $Q$ and distribution $\pi$ is reversible if $\pi$ and $Q$ satisfy the detailed balance equations

$$\pi(i) \cdot q(i, j) = \pi(j) \cdot q(j, i) \quad \forall i, j \in S.$$ 

Show that if $\pi$ is a reversible distribution for a CTMC, then $\pi$ is also a stationary distribution for the chain, and moreover the embedded jump chain is also reversible. Remark: the converse is also true — the CTMC is reversible if and only if the embedded chain is reversible.
3. **Jump Chain Stationary Distribution**

Use properties of transient states and the jump chain to find the stationary distribution of this CTMC.

![Diagram of a jump chain with states N, E, W, S and transition rates 1, 1, 2, 3, 4.](image-url)