EECS 126 Probability and Random Processes
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Midterm 1

Last Name	First Name	SID

Left Neighbor First and Last Name	**Right Neighbor** First and Last Name

Rules.

- **Write in your SID on every page to receive 1 point.**
- **Unless otherwise stated, all your answers need to be justified and your work must be shown. Answers without sufficient justification will get no credit.**
- You have 10 minutes to read the exam and 70 minutes to complete the exam. (DSP students with $X\,\%$ time accommodation should spend $10 \cdot X\,\%$ time on reading and $70 \cdot X\,\%$ time on completing the exam).
- This exam is closed-book. You may reference one double-sided handwritten sheet of paper. No calculator or phones allowed.
- Collaboration with others is strictly prohibited. If you are caught cheating, you may fail the course and face disciplinary consequences.

Problem	out of
SID	1
Problem 1	41
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Problem 5	22
Total	126

1 Potpourri [41 points]

(a) **Beat Stanford [9 points]**

Cal and Stanford are playing a best-of-three basketball match. The games are played at home, away and home respectively from the perspective of Cal. We know that Cal has a $\frac{2}{3}$ probability of winning at our home court and a $\frac{1}{2}$ probability of winning at an away court, independent of all other games. Given that Cal wins the best-of-three, what is the probability that all three games are needed to decide the champion? (In a best-of-three, the teams play a total of at most three games, and whoever wins two games is the champion. If a team wins both of the first two games, there is no need to play the third game.)

(b) **Working Monkeys [7 points]**

Sohom and his army of monkeys are working on writing a new edition of the EECS 126 textbook. At each timestep from $i = 0, 1, 2, \ldots$, Sohom writes down a random character X_i drawn uniformly at random from the alphabet $\{\text{`}a\text{'}, \text{`}b\text{'}, \text{`}c\text{'}, \ldots, \text{`}z\text{'}\}$, independent of all other events. There are 26 letters in the English alphabet. Show that with probability 1, Sohom writes down an 'a' eventually. In other words, if A_i is the event $\{X_i = \text{`}a\text{'}\}$, show that $\lim_{n\to\infty} \mathrm{P}(\bigcup_{i=1}^{n} A_i) = 1$.

(c) **Bad Beats [10 points]**

Poker extraordinaire Aadil is looking to see the odds that he loses money at the table tonight. He has been playing so dominantly that he will need to lose at least 90% of the n remaining rounds to lose money. The probability that Aadil loses each round is $\frac{1}{2}$, independent of all other rounds. Using Chebyshev's Inequality, show that the probability of Aadil losing money is at most $\frac{25}{32n}$.

(d) **Order! [9 points]**

Consider the sequence of i.i.d. random variables $X_1, X_2, \ldots, X_n$ where $X_i \sim \text{Uniform}[0,1]$ for all i. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be their order statistics, where $X_{(k)}$ is the kth smallest element of the sequence. What is the joint PDF of $X_{(1)}$ and $X_{(n)}$, namely $f_{X_{(1)},X_{(n)}}(x_1, x_n)$ for $x_1 \leq x_n$?

(e) **Encrypted Message [6 points]**

We are trying to send information across a channel using the fountain codes scheme **that we used in lab**. There are 4 data chunks, A, B, C, D, and Reina is able to successfully decode all of them using 4 packets, $(A, B), (A, B, D)$, (C) and a packet P whose content is unknown to us. What are all possible contents of P? Express your answer in tuples.

2 Café 126 [16 points]

When the popular EECS126 Café opens for business at 1:26PM, there is already a queue of $N \sim \text{Poisson}(\mu)$ people waiting outside. The café serves one person at a time, and the ith person's service time is X_i, where $X_i \sim \text{Exponential}(\lambda)$ for all i, independent of each other and N. Let $Y = \sum_{i=1}^{N} X_i$ be the total time needed to serve all customers that have been waiting in queue before Café 126 opens.

(a) What is $\mathrm{E}[Y]$?

(b) What is $\mathrm{var}(Y)$?

3 Competing Geometrics [22 points]

Let $X_1 \sim \text{Geometric}(p_1)$ and $X_2 \sim \text{Geometric}(p_2)$ be two independent random variables.

(a) Let $Y = \min(X_1, X_2)$. Find the distribution of Y.

(b) Find $\text{P}(X_1 < X_2)$.

(c) Let $Z = \max(X_1, X_2) - \min(X_1, X_2)$. Write $M_Z(s)$ in terms of $M_{X_1}(s)$ and $M_{X_2}(s)$, where $M_Z(s)$ is the moment generating function of Z. *Hint: condition Z on each of the three events $\{X_1 < X_2\}$, $\{X_1 > X_2\}$ and $\{X_1 = X_2\}$.*

4 Rounded Exponentials [24 points]

Let $X \sim \text{Exponential}(\lambda)$, and let Y be the value of X rounded to the nearest integer. Namely,

$$Y = \begin{cases} \lfloor X \rfloor & \text{if } X - \lfloor X \rfloor < 0.5 \\ \lceil X \rceil & \text{if } X - \lfloor X \rfloor \geq 0.5 \end{cases}$$

where $\lfloor x \rfloor = \max\{k : k \leq x, k \in \mathbb{Z}\}$ is the floor function and $\lceil x \rceil = \min\{k : k \geq x, k \in \mathbb{Z}\}$ is the ceiling function.

(a) Let $\mathbb{1}_{\{X - \lfloor X \rfloor < 0.5\}}$ be the indicator random variable for the event that X is rounded down. Identify the distribution name of $\mathbb{1}_{\{X - \lfloor X \rfloor < 0.5\}}$ along with its parameter(s). Then, find its expectation and variance.

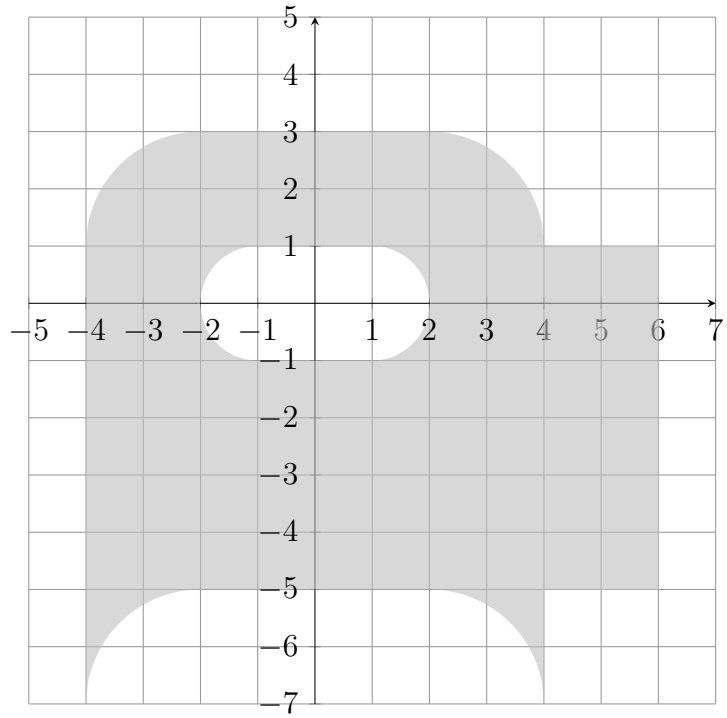
(b) Identify the distribution name of $\lceil X \rceil$ along with its parameter(s). Then, find its expectation and variance.

(c) Write Y in terms of $\mathbb{1}_{\{X - \lfloor X \rfloor < 0.5\}}$ and $\lceil X \rceil$. Then, find the expectation and variance of Y in terms of $\text{E}\left[\mathbb{1}_{\{X - \lfloor X \rfloor < 0.5\}}\right]$, $\text{var}\left(\mathbb{1}_{\{X - \lfloor X \rfloor < 0.5\}}\right)$, $\text{E}\left[\lceil X \rceil\right]$ and $\text{var}\left(\lceil X \rceil\right)$.

Here is some extra workspace for you. If you wish to use it, please indicate here which problem it is for and indicate on the original problem page that it leads to here.

5 Graphical Density? [22 points]

Consider the following shaded region in the x-y plane with uniform probability density ρ. The total area of the region is $72 - \pi$.



(a) **S**how your work: find ρ. You may leave answers to later subparts in terms of ρ.

(b) **U**sing ρ in part (a), find $\mathrm{E}[X]$ and $\mathrm{E}[X \mid Y = y]$ for $|y| \geq 1$.

(c) **S**till showing your work, find $f_X(x)$ for $|x| \geq 2$ and $f_{Y|X}(y \mid x)$ for $|x| \geq 2$.

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