EECS 126 Probability and Random Processes — UC Berkeley, Fall 2022

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Midterm 2

Last Name	First Name	SID

Left Neighbor First and Last Name	**Right Neighbor** First and Last Name

***Rules*.**

- **Unless otherwise stated, all your answers need to be justified and your work must be shown. Answers without sufficient justification will get no credit.**
- You have 10 minutes to read the exam and 70 minutes to complete the exam. (DSP students with $X\%$ time accommodation should spend $10 \cdot X\%$ time on reading and $70 \cdot X\%$ time on completing the exam).
- This exam is closed-book. You may reference one double-sided handwritten sheet of paper. No calculator or phones allowed.
- Collaboration with others is strictly prohibited. If you are caught cheating, you may fail the course and face disciplinary consequences.

Problem	out of
SID	1
Problem 1	30
Problem 2	15
Problem 3	15
Problem 4	15
Problem 5	20
Problem 6	30
Total	126

 Student ID (write on every page for 1 point): ____________

1 Scheduling Conflict [30 points]

Oh no, EECS 126 and CS 170 are having Homework Parties at Cory Courtyard at the same time! EECS 126 and CS 170 students arrive at Cory Courtyard independently according to two Poisson processes with rates λ_{126} and λ_{170} respectively. For simplicity, assume no student is taking the two classes at the same time.

(a) Let T_3 be the time that the third student arrives. What is $\mathrm{E}[T_3]$?

Andy walks by Cory Courtyard at time t and sees there are three students.

(b) What is the expected time between the last student arrival before Andy and the next student arrival after Andy?

(c) What is the probability that Andy sees more EECS 126 students than CS 170 students when he walks by?

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2 IKEA Shopping [15 points]

At a popular plush toy and furniture store, Axel observes that customers enter according to a Poisson process with rate λ, and *each customer* spends i.i.d. Exponential(μ) time in the store before leaving.

(a) Draw the state transition diagram for a CTMC to model the number of customers in the store.

(b) What is the stationary distribution of the CTMC? For simplicity, you do not need to normalize the distribution but can choose to leave it in terms of C, a normalizing constant.

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3 An Experimental Algorithm [15 points]

Recall the Metropolis-Hastings algorithm, which allows us to generate samples from some distribution $\pi(x)$ by simulating an appropriate Markov chain. As in homework and lab, suppose we have access to unnormalized density $f(x) = C\pi(x)$ for unknown C and proposal distribution $g(x, \cdot)$, but we are experimenting with different choices of the acceptance function $A(x, y)$. For each of the following possible acceptance functions, determine whether or not samples from the resulting Metropolis-Hastings variant will follow the desired distribution $\pi(x)$ and briefly explain why.

Hint: At each step of the algorithm, at state x, we propose a candidate state y with probability $g(x, y)$ and accept it with probability $A(x, y)$, so $P(x, y) = g(x, y)A(x, y)$. Your goal is to determine whether the transition probabilities make the Markov Chain reversible under $\pi(x)$.

(a) $A(x, y) = \min\left\{1, \frac{f(y)g(y,x)}{f(x)g(x,y)}\right\}$

(b) $A(x, y) = \min\left\{\frac{1}{2}, \frac{1}{2} \cdot \frac{f(y)g(y,x)}{f(x)g(x,y)}\right\}$

(c) $A(x, y) = \max\left\{\frac{9}{10}, \min\left\{1, \frac{f(y)g(y,x)}{f(x)g(x,y)}\right\}\right\}$

4 Brink of Ruin [15 points]

Alex, realizing his wallet is empty, finds himself entered into the following game. He starts with \$1. Then, on each turn, he flips a fair coin, winning one dollar if it comes up heads and losing one dollar otherwise. He also wins a *gambler's token* for every turn where he has only \$1, including the start. The game ends when he has no more money or has \$100 in total. What is the expected number of gambler's tokens Alex has when the game ends?

Hint: consider applying first-step analysis to β_i, the expected *number of visits* to state \$1 starting from $\$i$. To solve the recurrence relation that you find, use the fact that if b is the average of a and c, then $c - b = b - a$ is a common difference.

5 Entropic Maneuvers [20 points]

Consider two (potentially dependent) random variables X and Y that each takes value in the set $\{1, 2, \ldots, n\}$. Let $E = \mathbb{1}_{\{X \neq Y\}}$ be the indicator random variable for the event that X and Y are not equal, and let $p = \mathrm{P}(X \neq Y) = \mathrm{P}(E = 1)$.

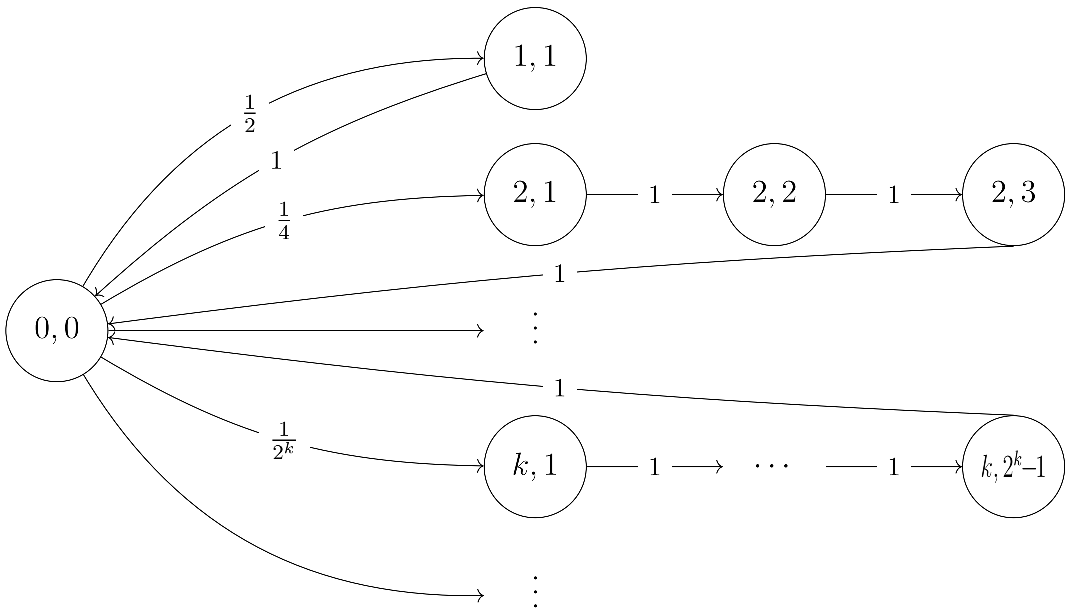
(a) Show that $H(X, E \mid Y) = H(X \mid Y)$.

(b) Using the previous part, show that $H(X \mid Y) \leq p \log_2(n-1) + H(E)$.

Hint: Use the fact that the uniform distribution on a set of k elements has an entropy of $\log_2 k$ and has the maximum entropy among all distributions on the set.

6 Bot on a Stroll [30 points]

The EECS 126 Bot is taking a walk on a Markov chain with state space $\mathbb{N} \times \mathbb{N}$, starting from state $(0, 0)$, as shown by the graph below.



From state $(0, 0)$, the bot chooses "path k" with probability 2^{-k} for $k = 1, 2, \ldots$ Each path k contains $2^k - 1$ states, which the bot will travel through in sequence then return to $(0, 0)$ deterministically.

(a) Is this Markov chain irreducible? Justify your answer.

(b) What is the period of this Markov chain?

(c) What is the expected time to return to state $(0, 0)$?

(d) Is this Markov chain positive recurrent, null recurrent, or transient? Justify your answer.

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Here is some extra workspace for you. If you wish to use it, please indicate which problem it is for and indicate on the original problem page that it leads to here.