1. Bus Arrivals at Cory Hall

Starting at time 0, the 52 line makes stops at Cory Hall according to a Poisson process of rate $\lambda$. Students arrive at the stop according to an independent Poisson process of rate $\mu$. Every time the bus arrives, all students waiting get on.

a. Given that the interarrival time between bus $i-1$ and bus $i$ is $x$, find the distribution for the number of students entering the $i$th bus. Here, $x$ is a given number, not a random quantity.

b. Given that a bus arrived at 9:30 AM, find the distribution for the number of students that will get on the next bus.

c. Find the distribution of the number of students getting on the next bus to arrive after 9:30 AM, assuming that time 0 was infinitely far in the past.

Solution:

a. The student arrival process is independent of the bus arrival process, so the number of students arrivals in this time interval of length $x$ is Poisson with parameter $\mu x$.

b. Let us consider the merged process of student and bus arrivals, which has rate $\lambda + \mu$. Each arrival for the combined process is a bus with probability $p := \frac{\lambda}{\lambda + \mu}$ and a student with probability $\frac{\mu}{\lambda + \mu}$, and these “choices” can be treated as i.i.d. Bernoulli trials. Thus, starting right after the arrival at 9:30 AM, the number of combined arrivals until we see a bus arrival for the first time is Geometric with parameter $p$. If $N$ is the number of students entering the next bus after 9:30 AM, then for $n \in \mathbb{N}$,

$$P(N = n) = \left( \frac{\mu}{\lambda + \mu} \right)^n \frac{\lambda}{\lambda + \mu}.$$ 

Alternate solution. Let $T \sim \text{Exponential}(\lambda)$ be the interarrival time between the 9:30 AM bus arrival and the next bus, and let $N$ be the number of students who arrived between 9:30 AM and 9:30 AM + $T$. We know that $N \mid T = t \sim \text{Poisson}(\mu t)$, so by the law of total probability,

$$P(N = n) = \int_0^\infty P(N = n \mid T = t) \cdot f_T(t) \, dt$$

$$= \int_0^\infty \frac{(\mu t)^n}{n!} e^{-\mu t} \cdot \lambda e^{-\lambda t} \, dt$$

$$= \frac{\mu^n}{n!} \frac{\lambda}{\lambda + \mu} \int_0^\infty t^n (\lambda + \mu) e^{-(\lambda + \mu) t} \, dt$$

$$= \frac{\mu^n}{n!} \frac{\lambda}{\lambda + \mu} \mathbb{E}(\text{Exponential}(\lambda + \mu)^n)$$
\[
\frac{\mu^n}{n!} \frac{\lambda}{\lambda + \mu} \frac{n!}{(\lambda + \mu)^n}
\]

which simplifies to the same answer as above.

c. This subpart is based on the random incidence property of the Poisson process. In part b, we found the number of future student arrivals \(N_2\) before the next bus; now, we seek the sum of \(N_2\) and the number of students \(N_1\) already waiting at the bus stop at 9:30. We observe that by memorylessness, \(N_1\) and \(N_2\) are independent. We also make the key observation that \(N_1\) and \(N_2\) are equal in distribution: if we consider the Poisson process backwards in time, \(N_1\) is the number of students waiting from 9:30 until the “next” (previous) bus arrival. The assumption that time 0 is infinitely far in the past guarantees at least one such arrival exists, because \(\lim_{t \to \infty} N_t \geq 1\) a.s. Now, we can find the distribution of \(N = N_1 + N_2\) by convolution: for \(n \in \mathbb{N}\),

\[
P(N = n) = \sum_{k=0}^{n} P(N_1 = k) \cdot P(N_2 = n - k)
\]

\[
= \sum_{k=0}^{n} \left( \frac{\mu}{\lambda + \mu} \right)^k \left( \frac{\lambda}{\lambda + \mu} \right) \cdot \left( \frac{\mu}{\lambda + \mu} \right)^{n-k} \left( \frac{\lambda}{\lambda + \mu} \right)
\]

\[
= (n + 1) \left( \frac{\mu}{\lambda + \mu} \right)^n \left( \frac{\lambda}{\lambda + \mu} \right)^2
\]

Alternate solution. Let \(T_1\) be the time from the bus arrival right before 9:30 until 9:30, and let \(T_2\) be the time from 9:30 until the next bus arrival. As above, we know that an arrival before 9:30 exists by the assumption that time 0 is infinitely far in the past. Now, \(T_1\) and \(T_2\) are i.i.d. Exponential(\(\lambda\)), so we know that \(T = T_1 + T_2 \sim \text{Erlang}(2, \lambda)\). If \(N\) is the number of students who get on the next bus after 9:30, then \(N \mid T = t \sim \text{Poisson}(\mu t)\). By the law of total probability,

\[
P(N = n) = \int_0^{\infty} \frac{(\mu t)^n}{n!} e^{-\mu t} \cdot \lambda^2 t e^{-\lambda t} dt
\]

\[
= \frac{\mu^n}{n!} \frac{\lambda^2}{\lambda + \mu} \int_0^{\infty} t^{n+1}(\lambda + \mu)e^{-(\lambda+\mu)t} dt
\]

\[
= \frac{\mu^n}{n!} \frac{\lambda^2}{\lambda + \mu} (n + 1)!
\]

which gives the same answer as the approach above.
2. Lazy Server

Customers arrive at a queue at the times of a Poisson process with rate \( \lambda \). The queue is in a service facility with infinite capacity, in which there is an infinitely powerful but lazy server who visits the facility at the times of a Poisson process with rate \( \mu \). These two processes are independent. When the server visits the facility, it instantaneously serves all the customers in the queue, then immediately leaves. In other words, at any time, the only customers waiting in the queue are those who arrived after the server’s most recent visit.

a. Model the queue length as a CTMC, and find its stationary distribution.

b. Supposing that the CTMC is at stationarity, find the mean number of customers waiting in the queue at any given time.

Solution:

a. We can model the queue length as a continuous-time Markov chain on the state space \( S = \mathbb{N} \). The rate at which a customer arrives is \( \lambda \), and the rate at which the server arrives is \( \mu \), so the rates are \( q(i, i+1) = \lambda \) for \( i \in \mathbb{N} \) and \( q(i, 0) = \mu \) for \( i \in \mathbb{Z}^+ \). Now, the balance equation for state \( i \in \mathbb{Z}^+ \) reads \( \lambda \cdot \pi(i-1) = (\lambda + \mu) \cdot \pi(i) \), a recurrence relation whose base case we can find by

\[
\sum_{i \in \mathbb{N}} \pi(i) = \sum_{i \in \mathbb{N}} \left( \frac{\lambda}{\lambda + \mu} \right)^i \pi(0) = \frac{1}{1 - \frac{\lambda}{\lambda + \mu}} \pi(0) = \frac{\lambda + \mu}{\mu} \pi(0) = 1.
\]

With \( \pi(0) = \frac{\mu}{\lambda + \mu} \), the stationary distribution is given by

\[
\pi(i) = \frac{\mu}{\lambda + \mu} \left( \frac{\lambda}{\lambda + \mu} \right)^i.
\]

b. If \( X \) is a random variable with \( P(X = i) = \pi(i) \) for all \( i \in S \), then we see that

\[
X + 1 \sim \text{Geometric} \left( \frac{\mu}{\lambda + \mu} \right).
\]

Thus \( E(X) = \frac{\lambda + \mu}{\mu} - 1 = \frac{\lambda}{\mu} \). One possible interpretation of this fact is that \( \frac{1}{\mu} \) is the mean amount of time a customer spends in the system.
3. M/M/2 Queue

A queue has Poisson arrivals with rate $\lambda$ and two servers with i.i.d. Exponential($\mu$) service times. The two servers work in parallel: when there are at least two customers in the queue, two are being served; when there is only one customer, only one server is active. Let $X_t$ be the number of customers either in the queue or in service at time $t$.

a. Argue that the process $(X_t)_{t \geq 0}$ is a Markov process, and draw its state transition diagram.

b. Find the range of values of $\mu$ for which the Markov chain is positive recurrent. For this range of values, calculate the stationary distribution of the Markov chain.

Solution:

a. The queue length is a MC: customer arrivals are independent of the current number of customers in the queue, and departures only depend on the current number of customers being served. Also, even when $k = 1$ or 2 customers are being served, the completion of their services are independent of one another. Finally, when $k = 2$, even if one of the customers has been completely served, the other customer has the same service time distribution as before, because the Exponential distribution is memoryless.

b. It suffices to solve the detailed balance equations

$$
\pi(1) = \frac{\lambda}{\mu}\pi(0)
$$

$$
\pi(i + 1) = \frac{\lambda}{2\mu}\pi(i), \quad i \in \mathbb{Z}^+.
$$

Iterating these recurrences yields the following expression for the stationary distribution. We can find the base case $\pi(0)$ as the stationary distribution must normalize:

$$
\sum_{i=0}^{\infty} \pi(i) = \pi(0) + \pi(0) \cdot \frac{\lambda}{\mu} \sum_{i=1}^{\infty} \left( \frac{\lambda}{2\mu} \right)^{i-1} = 1.
$$

This series converges iff $\lambda < 2\mu$, in which case the Markov chain is positive recurrent. For $\mu$ in this range, we find the stationary distribution

$$
\pi(0) = \frac{2\mu - \lambda}{2\mu + \lambda}
$$

$$
\pi(i) = \frac{2\mu - \lambda}{2\mu + \lambda} \left( \frac{\lambda}{\mu} \right) \left( \frac{\lambda}{2\mu} \right)^{i-1}, \quad i \in \mathbb{Z}^+.
$$
4. Frogs

Three frogs are playing near a pond. When they are in the sun, they get too hot and jump in the lake at rate 1. When they are in the lake, they get too cold and jump onto land at rate 2. The rates here refer to those of the Exponential distribution. Let $X_t$ be the number of frogs in the sun at time $t \geq 0$.

a. Find the stationary distribution of $(X_t)_{t \geq 0}$.

b. Find the answer to part a again, this time using the observation that the three frogs are independent two-state Markov chains.

Solution:

a. Let the states $S = \{0, 1, 2, 3\}$ be the number of frogs in the sun. The Markov chain has $\lambda_0 = 6$, $\lambda_1 = 4$, $\lambda_2 = 2$, $\mu_3 = 3$, $\mu_2 = 2$, and $\mu_1 = 1$, where $\lambda_i$ and $\mu_i$ are the rates of jumping forwards and backwards respectively from state $i$. Using detailed balance, we compute the stationary distribution to be

$$\pi = \frac{1}{27} [1 \quad 6 \quad 12 \quad 8].$$

b. The individual frogs follow independent Markov chains, each with stationary distribution

$$\pi = \frac{1}{3} [1 \quad 2].$$

The stationary probability of being in state $i \in S$ is therefore

$$\mathbb{P}(X_t = i) = \binom{3}{i} \left( \frac{1}{3} \right)^{3-i} \left( \frac{2}{3} \right)^i.$$
5. Jukes–Cantor Model

In this question, we consider a CTMC model for the evolution of DNA over time. Consider a CTMC \((X_t)_{t \geq 0}\) on the states \(\mathcal{X} := \{A, C, G, T\}\) with transition rate matrix

\[
Q = \begin{pmatrix}
-3\lambda & \lambda & \lambda & \lambda \\
\lambda & -3\lambda & \lambda & \lambda \\
\lambda & \lambda & -3\lambda & \lambda \\
\lambda & \lambda & \lambda & -3\lambda
\end{pmatrix},
\]

where \(\lambda > 0\). That is, all edges in the following transition diagram have rate \(\lambda\):

\[A \quad T \quad \quad \quad \quad \quad \quad \quad C \quad G\]

For \(x, y \in \mathcal{X}\), what is \(P_t(x, y) := \mathbb{P}(X_t = y \mid X_0 = x)\)? What happens as \(t \to \infty\)?

*Hint:* consider adding a self-loop with rate \(\lambda\) to each of the four states. Does this change the behavior of the CTMC? Then, condition on whether a transition occurs or not.

**Solution:** An equivalent way of describing the CTMC is as follows. Starting from the initial state, the chain makes a jump with rate \(4\lambda\), and it jumps to any of the four states in \(\mathcal{X}\) with probability \(\frac{1}{4}\). Note that the chain is at stationarity after one jump, where the stationary distribution is, by symmetry, the uniform distribution

\[
\pi = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}.
\]

Thus, with probability \(e^{-4\lambda t}\), the chain has not yet made a jump by time \(t\), in which case it is still at its starting state; otherwise, with probability \(1 - e^{-4\lambda t}\), the chain is equally likely to be in any of the four states.

\[
P_t(x, y) = \begin{cases} 
e{4\lambda t} + \frac{1}{4}(1 - e^{-4\lambda t}), & x = y \\ \frac{1}{4}(1 - e^{-4\lambda t}), & x \neq y. \end{cases}
\]

As \(t \to \infty\), the chain approaches stationarity starting from any initial state:

\[P_t(x, y) \to \frac{1}{4} \quad \forall x, y \in \mathcal{S}.\]
6. **Two-Server System**

A company has two servers. The second server is a backup in case the first server fails, so only one server is ever used at a time. When a server is running, the time until it breaks down is Exponentially distributed with rate $\mu$. When a server is broken, it is taken to the repair shop. The repair shop can only fix one server at a time, and its repair time is Exponentially distributed with rate $\lambda$. Find the long-run probability that no servers are operational.

**Solution:** We can model the number of operational servers as a continuous-time Markov chain on the state space $\{0, 1, 2\}$. We have transition rate matrix

$$
Q = \begin{bmatrix}
-\lambda & \lambda & 0 \\
\mu & -(\lambda + \mu) & \lambda \\
0 & \mu & -\mu
\end{bmatrix}
$$

and state transition diagram

Now, we write down the balance equations.

$$
\lambda \pi(0) = \mu \pi(1) \\
(\lambda + \mu) \pi(1) = \lambda \pi(0) + \mu \pi(2) \\
\mu \pi(2) = \lambda \pi(1) \\
1 = \pi(0) + \pi(1) + \pi(2).
$$

Eliminating $\pi(2) = \frac{\lambda}{\mu} \pi(1)$, we have

$$
\mu \pi(1) = \lambda \pi(0) \\
1 = \pi(0) + \left(1 + \frac{\lambda}{\mu}\right) \pi(1),
$$

after which we can eliminate $\pi(1) = \frac{\lambda}{\mu} \pi(0)$ to find

$$
\pi(0) = \frac{1}{1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2}.
$$

This is the long-run probability that we will be in state 0, i.e. there are no operational servers.