1. **Midterm**

Solve again the midterm problems which you got incorrect. Please demonstrate understanding of the questions without simply copying the solutions.
2. Uniformization

Consider the CTMC with state space \( \{1, 2, 3\} \) and rate matrix

\[
Q = \begin{bmatrix}
-4 & 1 & 3 \\
0 & -3 & 3 \\
1 & 1 & -2
\end{bmatrix}.
\]

a. Let \( \lambda = 5 \). Find the transition probability matrix of the uniformized chain.

b. Describe what \( \lambda \) represents. Why must \( \lambda \geq \max_i q_i \)?

c. Intuitively, why does the probability of a self-jump increase when \( \lambda \) increases?
3. Connected Random Graph

We start with the empty graph on \( n \) vertices. Iteratively, we add an undirected edge \( \{u, v\} \), chosen uniformly at random from the edges that are not yet present in the graph, until the graph is connected. Let \( X \) be the random variable equal to the total number of edges in the graph. Show that \( E(X) \in O(n \log n) \).

**Hint:** consider \( X_k \), the number of edges added while there are \( k \) connected components until there are \( k - 1 \) connected components. Do not try to calculate \( E(X_k) \), as an upper bound is enough.
4. Isolated Vertices

Consider an Erdős–Rényi random graph $G(n, p(n))$, where $n$ is the number of vertices and $p(n)$ is the probability that any specific edge appears in the graph. Let $X_n$ be the number of isolated vertices in $G(n, p(n))$. Show that

$$\mathbb{E}(X_n) \xrightarrow{n \to \infty} \begin{cases} 
\infty, & p(n) \ll \frac{\ln n}{n} \\
\exp(-c), & p(n) = \frac{(\ln n) + c}{n} \\
0, & p(n) \gg \frac{\ln n}{n}.
\end{cases}$$

The asymptotic notation $p(n) \ll f(n)$ means that $\frac{p(n)}{f(n)} \to 0$ as $n \to \infty$. Also show that in the case of $p(n) \gg \frac{\ln n}{n}$, we have $X_n \to 0$ in probability as well.

*Hint:* from Taylor series expansion, one can show that $\ln(1 + x) < x$ for any $x$, and moreover $\ln(1 + x) \approx x$ when $x$ is small.
5. **Subcritical Forest**

Consider a random graph $G(n, p(n))$ with $p(n) = o\left(\frac{1}{n}\right)$ (this is called the subcritical phase).

a. Let $X_n$ be the number of cycles in the graph. Show that $E(X_n) \to 0$.

b. Show that the probability that $G(n, p(n))$ is a forest tends to 1 as $n \to \infty$. A forest is a possibly disconnected graph which contains no cycles.