1. Gaussian Hypothesis Testing

Consider a hypothesis testing problem that if $X = 0$, you observe a sample of $\mathcal{N}(\mu_0, \sigma^2)$, and if $X = 1$, you observe a sample of $\mathcal{N}(\mu_1, \sigma^2)$, where $\mu_0, \mu_1 \in \mathbb{R}$, $\sigma^2 > 0$. Find the Neyman-Pearson test for false alarm $\beta \in (0, 1)$, that is, $\mathbb{P}(\hat{X} = 1 \mid X = 0) \leq \beta$. 
2. BSC Hypothesis Testing

Consider a BSC with some error probability $\epsilon \in [0.1, 0.5)$. Given $n$ inputs and outputs $(x_i, y_i)$ of the BSC, solve a hypothesis problem to detect that $\epsilon > 0.1$ with a probability of false alarm at most equal to 0.05. Assume that $n$ is very large and use the CLT.

*Hint:* The null hypothesis is $\epsilon = 0.1$. The alternate hypothesis is $\epsilon > 0.1$, which is a composite hypothesis (this means that under the alternate hypothesis, the probability distribution of the observation is not completely determined; compare this to a simple hypothesis such as $\epsilon = 0.3$, which does completely determine the probability distribution of the observation). The Neyman-Pearson Lemma we learned in class applies for the case of a simple null hypothesis and a simple alternate hypothesis, so it does not directly apply here.

To fix this, fix some specific $\epsilon' > 0.1$ and use the Neyman-Pearson Lemma to find the optimal hypothesis test for the hypotheses $\epsilon = 0.1$ vs. $\epsilon = \epsilon'$. Then, argue that the optimal decision rule does not depend on the specific choice of $\epsilon'$; thus, the decision rule you derive will be simultaneously optimal for testing $\epsilon = 0.1$ vs. $\epsilon = \epsilon'$ for all $\epsilon' > 0.1$. 


3. Hypothesis Testing for Uniform Distribution

Assume that

- If $X = 0$, then $Y \sim \text{Uniform}[-1, 1]$.
- If $X = 1$, then $Y \sim \text{Uniform}[0, 2]$.

Using the Neyman-Pearson formulation of hypothesis testing, find the optimal randomized decision rule $r : [-1, 2] \rightarrow \{0, 1\}$ with respect to the criterion

$$\min_{\text{randomized } r : [-1, 2] \rightarrow \{0, 1\}} \mathbb{P}(r(Y) = 0 \mid X = 1)$$

$$\text{s.t. } \mathbb{P}(r(Y) = 1 \mid X = 0) \leq \beta,$$

where $\beta \in [0, 1]$ is a given upper bound on the false positive probability.

(Note: We will be following the notation used in Walrand and lecture where the probability of false alarm (PFA) is bounded by $\beta$ as opposed to $\alpha$ used in the course notes.)
4. Photodetector LLSE

Consider a photodetector in an optical communications system that counts the number of photons arriving during a certain interval. A user conveys information by switching a photon transmitter on or off. Assume that the probability of the transmitter being on is \( p \). If the transmitter is on, the number of photons transmitted over the interval of interest is a Poisson random variable \( \Theta \) with mean \( \lambda \), and if it is off, the number of photons transmitted is 0. Unfortunately, regardless of whether or not the transmitter is on or off, photons may be detected due to “shot noise”. The number \( N \) of detected shot noise photons is a Poisson random variable \( N \) with mean \( \mu \), independent of the transmitted photons. Let \( T \) be the number of transmitted photons and \( D \) be the number of detected photons. Find \( L[T \mid D] \).
5. **Gaussian LLSE**

The random variables $X, Y, Z$ are i.i.d. $\mathcal{N}(0, 1)$.

(a) Find $L[X^2 + Y^2 | X + Y]$.
(b) Find $L[X + 2Y | X + 3Y + 4Z]$.
(c) Find $L[(X + Y)^2 | X - Y]$. 
6. Projections

The following exercises are from the note on the Hilbert space of random variables. See the notes for some hints.

(a) Let $\mathcal{H} := \{X : X$ is a real-valued random variable with $\mathbb{E}[X^2] < \infty\}$. Prove that $\langle X, Y \rangle := \mathbb{E}[XY]$ makes $\mathcal{H}$ into a real inner product space.  

(b) Let $U$ be a subspace of a real inner product space $V$ and let $P$ be the projection map onto $U$. Prove that $P$ is a linear transformation.

(c) Suppose that $U$ is finite-dimensional, $n := \dim U$, with basis $\{v_i\}_{i=1}^n$. Suppose that the basis is orthonormal. Show that $Py = \sum_{i=1}^n \langle y, v_i \rangle v_i$. (Note: If we take $U = \mathbb{R}^n$ with the standard inner product, then $P$ can be represented as a matrix in the form $P = \sum_{i=1}^n v_i v_i^T$.)

---

1To be perfectly correct, it is possible for $X \neq 0$ but $\mathbb{E}[X^2] = 0$; this occurs if $X = 0$ with probability 1. To fix this, we need to define two random variables $X$ and $Y$ to be equal if $\mathbb{P}(X = Y) = 1$. In other words, we consider equivalence classes of random variables, defined by the relation $\equiv$. With this definition, then if $X \neq 0$ we do indeed have $\mathbb{E}[X^2] > 0$. 