1. Joint Occurrence

You know that at least one of the events $A_i$, $i = 1, \ldots, n$, is certain to occur, but certainly no more than two occur. $n$ is an integer $\geq 2$. Show that if the probability of occurrence of any single event is $p$, and the probability of joint occurrence of any two distinct events is $q$, we have $p \geq \frac{1}{n}$ and $q \leq \frac{2}{n(n-1)}$.

**Solution:** By the union bound, since

$$1 = P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) = np,$$

we see that $p \geq \frac{1}{n}$. Now we observe that the events $A_i \cap A_j$ for $i < j$, $i, j \in \{1, \ldots, n\}$ are pairwise disjoint, so by finite additivity,

$$1 \geq P\left(\bigcup_{i<j} A_i \cap A_j\right) = \sum_{i<j} P(A_i \cap A_j) = \binom{n}{2} q,$$

so $q \leq \frac{1}{\binom{n}{2}} = \frac{2}{n(n-1)}$. 


2. Packet Routing

Consider a system with \( n \) inputs and \( n \) outputs. At each input, a packet appears independently with probability \( p \). If a packet appears, it is destined for one of the \( n \) outputs uniformly randomly, independently of the other packets.

a. Let \( X \) denote the number of packets destined for the first output. What is the distribution of \( X \)?

b. What is the probability of a collision (that is, more than one packet heading to the same output)?

**Solution:**

a. The probability that there exists a packet at an input and the packet is destined for the first output is \( \frac{p}{n} \). By the independence over inputs, \( X \) has the binomial distribution \( \text{Binomial}(n, \frac{p}{n}) \).

b. Let \( C \) be the event of a collision and let \( N \) be the total number of packets in all inputs. Given that there are \( k \) packets at the input, where \( k \in \{0, \ldots, n\} \), the probability that there is no collision is

\[
P(C^c \mid N = k) = \frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-k+1}{n} = \frac{n!}{(n-k)!n^k}.
\]

The first packet can land anywhere, the second packet has to avoid the first packet, and can thus land in any of the \( n-1 \) remaining outputs, and so on. Therefore

\[
P(C) = 1 - P(C^c) = 1 - \sum_{k=0}^{\infty} P(C^c \mid N = k) \cdot P(N = k)
\]

\[
= 1 - \sum_{k=0}^{n} \frac{n!}{(n-k)!n^k} \binom{n}{k} p^k (1-p)^{n-k}.
\]
3. Compact Arrays

Consider an array of \( n \geq 1 \) entries, where each entry is chosen uniformly randomly from \([0, \ldots, 9]\). We want to make the array more compact by putting all of the nonzero entries together at the front of the array. For example, if we take the array

\[
\begin{bmatrix}
6 & 4 & 0 & 0 & 5 & 3 & 0 & 5 & 1 & 3
\end{bmatrix}
\]

and make it compact, we now have

\[
\begin{bmatrix}
6 & 4 & 5 & 3 & 5 & 1 & 3 & 0 & 0 & 0
\end{bmatrix}
\]

Let \( i \) be a fixed positive integer in \([1, \ldots, n]\). Suppose that the \( i \)th entry of the array is nonzero. (The array is indexed starting from 1.) Let \( X_i \) be the random variable equal to the index that the \( i \)th entry has been moved to after making the array compact. Calculate \( E(X_i) \) and \( \text{var}(X_i) \).

**Solution:** Let \( Y_j, j = 1, \ldots, i - 1 \), be the indicator that the \( j \)th entry of the original array is 0. Then the \( i \)th entry is moved backwards \( \sum_{j=1}^{i-1} Y_j \) positions, so

\[
E(X_i) = i - \sum_{j=1}^{i-1} E(Y_j) = i - \frac{i - 1}{10} = \frac{9i + 1}{10}.
\]

The variance is also straightforward to compute by the independence of the indicators \( Y_j \). We note that \( \text{var}(Y_j) = \frac{1}{10} \cdot \frac{9}{10} = \frac{9}{100} \), so

\[
\text{var}(X_i) = \text{var} \left( i - \sum_{j=1}^{i-1} Y_j \right) = \sum_{j=1}^{i-1} \text{var}(Y_j) = \frac{9(i - 1)}{100}.
\]
4. **Lightbulbs**

Consider an $n \times n$ array of switches. Each row $i$ of switches corresponds to a single lightbulb $L_i$, so that $L_i$ lights up if at least $i$ switches in row $i$ are flipped ON. All of the switches start in the OFF position, and each is flipped ON with probability $p$, independently of all others. What is the expected number of lightbulbs that will be lit up? Express your answer in closed form without any summations.

**Solution:** The number of switches ON in each row $i = 1, \ldots, n$ is a random variable $X_i \sim \text{Binomial}(n, p)$. We are interested in the following expectation, which is

$$E(1_{X_1 \geq 1} + 1_{X_2 \geq 2} + \cdots + 1_{X_n \geq n}) = \sum_{i=1}^{n} E(1_{X_i \geq i}) = \sum_{i=1}^{n} \mathbb{P}(X_i \geq i) = \sum_{i=1}^{n} \mathbb{P}(X \geq i)$$

by linearity. $X$ is any random variable with distribution $\text{Binomial}(n, p)$. Then, by the tail-sum formula, this is just $E(X) = np$. 

4
5. Expected Sorting Distance

Let \((a_1, \ldots, a_n)\) be a random permutation of \(\{1, \ldots, n\}\), so that it is equally likely to be any possible permutation. When sorting the list \((a_1, \ldots, a_n)\), the element \(a_i\) must move a distance of \(|a_i - i|\) places from its current position to reach the position in the sorted order. Find the expected total distance that the elements will have to be moved,

\[
E \left( \sum_{i=1}^{n} |a_i - i| \right)
\]

**Note:** To simplify your answer, you can use the formula

\[
\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.
\]

**Solution:** By the linearity of expectation, we have that

\[
E \left( \sum_{i=1}^{n} |a_i - i| \right) = \sum_{i=1}^{n} E(|a_i - i|).
\]

Because all of the permutations are equally likely, \(a_i\) is equally likely to be any number from 1 to \(n\). Thus

\[
E(|a_i - i|) = \frac{1}{n} \sum_{k=1}^{n-i} k
\]

\[
= \frac{1}{n} \sum_{k=1}^{n-i} k + \frac{1}{n} \sum_{k=1}^{i-1} k
\]

\[
= \frac{(n-i)(n-i+1) + (i-1)i}{2n}.
\]

Putting it all together, and using the closed-form formula for \(\sum_{k=1}^{n} k^2\), we obtain

\[
E \left( \sum_{i=1}^{n} |a_i - i| \right) = \frac{n^2 - 1}{3}.
\]
6. Poisson Properties

a. **Poisson merging.** Suppose $X$ and $Y$ are independent Poisson random variables with means $\lambda$ and $\mu$ respectively. Prove that $X + Y$ has the Poisson distribution with mean $\lambda + \mu$.

**Note:** It is not enough to use linearity of expectation to say that $X + Y$ has mean $\lambda + \mu$. You are asked to prove that the distribution of $X + Y$ is Poisson.

b. Given $X$ and $Y$ as above, what is the distribution of $X$ conditioned on $X + Y = z$, $z \in \mathbb{N}$?

c. **Poisson splitting.** Suppose that $Z \sim \text{Poisson}(\lambda)$. We flip $Z$ independent coins, each with probability of heads $p$. Let $X$ be the number of heads, and let $Y$ be the number of tails, so that $Z = X + Y$. Show that $X$ and $Y$ are independent Poisson random variables with means $\lambda p$ and $\lambda (1 - p)$ respectively.

**Solution:**

a. The distribution of the sum of two independent random variables is the convolution of their individual distributions. For $z \in \mathbb{N}$, we have

$$
P(X + Y = z) = \sum_{x=0}^{z} P(X = x) \cdot P(Y = z - x)
$$

$$
= \sum_{x=0}^{z} \frac{\lambda^x}{x!} e^{-\lambda} \cdot \frac{\mu^{z-x}}{(z-x)!} e^{-\mu}
$$

$$
= \frac{e^{-(\lambda+\mu)}}{z!} \sum_{x=0}^{z} \frac{z!}{x!(z-x)!} \lambda^x \mu^{z-x}
$$

$$
= \frac{e^{-(\lambda+\mu)}}{z!} \sum_{x=0}^{z} \binom{z}{x} \lambda^x \mu^{z-x}
$$

$$
= \frac{e^{-(\lambda+\mu)}}{z!} (\lambda + \mu)^z,
$$

which shows that $X + Y \sim \text{Poisson}(\lambda + \mu)$.

b. 

$$
P(X = x \mid X + Y = z) = \frac{P(X = x, X + Y = z)}{P(X + Y = z)}
$$

$$
= \frac{P(X = x) \cdot P(Y = z - x)}{\frac{(\lambda + \mu)^z}{z!} e^{-(\lambda+\mu)}}
$$

$$
= \frac{\lambda^x}{x!} e^{-\lambda} \cdot \frac{\mu^{z-x}}{(z-x)!} e^{-\mu} \left/ \left( \frac{(\lambda + \mu)^z}{z!} e^{-(\lambda+\mu)} \right) \right.
$$

$$
= \binom{z}{x} \left( \frac{\lambda}{\lambda + \mu} \right)^x \left( \frac{\mu}{\lambda + \mu} \right)^{z-x},
$$

which shows that $X \mid X + Y = z \sim \text{Binomial}(z, \frac{\lambda}{\lambda + \mu})$. 


c. By symmetry, it is enough to show that $X \sim \text{Poisson}(\lambda p)$.

\[
\mathbb{P}(X = x) = \sum_{z=x}^{\infty} \mathbb{P}(Z = z) \cdot \mathbb{P}(X = x \mid Z = z)
\]
\[
= \sum_{z=x}^{\infty} \frac{\lambda^z}{z!} e^{-\lambda} \binom{z}{x} p^x (1 - p)^{z-x}
\]
\[
= \frac{(\lambda p)^x}{x!} e^{-\lambda} \sum_{z=x}^{\infty} \frac{(\lambda(1 - p))^{z-x}}{(z-x)!}
\]
\[
= \frac{(\lambda p)^x}{x!} e^{-\lambda} e^{\lambda(1-p)}
\]
\[
= \frac{(\lambda p)^x}{x!} e^{-\lambda p}.
\]

We also need to show that $X$ and $Y$ are independent:

\[
\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x, Y = y \mid Z = x + y) \cdot \mathbb{P}(Z = x + y)
\]
\[
= \binom{x + y}{x} p^x (1 - p)^y \cdot \frac{\lambda^{x+y}}{(x+y)!} e^{-\lambda}
\]
\[
= \frac{(x+y)!}{x! \cdot y!} p^x (1 - p)^y \cdot \frac{\lambda^x}{(x)!} \frac{\lambda^y}{(y)!} e^{-\lambda p} e^{-\lambda(1-p)}
\]
\[
= \frac{(\lambda p)^x}{x!} e^{-\lambda p} \cdot \frac{(\lambda(1 - p))^y}{y!} e^{-\lambda(1-p)}
\]
\[
= \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y).
\]

Remark: These properties will be used extensively when we discuss the Poisson process.