1. **Convergence in Probability**

Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of i.i.d. random variables distributed uniformly in \([-1, 1]\). Show that the following sequences \((Y_n)_{n \in \mathbb{N}}\) converge in probability to some limit.

a. \(Y_n = \prod_{i=1}^{n} X_i\).

b. \(Y_n = \max\{X_1, \ldots, X_n\}\).

c. \(Y_n = (X_1^2 + \cdots + X_n^2)/n\).
2. Really Random Binomial

You have a binomial random variable \( X \sim \text{Binomial}(n, u) \). Unfortunately, you forget what the value of \( u \) is, so you assume that \( u \) is now a random variable \( U \sim \text{Uniform}([0, 1]) \), as you know that \( 0 \leq u \leq 1 \). Given that you sample from this binomial distribution and observe \( k \) successes, find the conditional distribution of \( U \) given \( X = k \).

**Hint**: Use MGFs to compute \( \Pr(X = k) \) instead of integrating the distribution directly. The binomial theorem might also be useful here. Also, recall the identity \( \sum_{i=0}^{n} s^i = \frac{1-s^{n+1}}{1-s} \).
3. Coupon Collector Bounds

Recall the coupon collector’s problem, in which there are \( n \) different types of coupons. Every box contains a single coupon, and we let the random variable \( X \) be the number of boxes bought until one of every type of coupon is obtained. The expected value of \( X \) is \( nH_n \), where \( H_n := \sum_{i=1}^{n} \frac{1}{i} \) is the harmonic number of order \( n \), which satisfies the inequality

\[
\ln n \leq H_n \leq \ln n + 1.
\]

a. Use Markov’s inequality in order to show that

\[
P(X > 2nH_n) \leq \frac{1}{2}.
\]

b. Use Chebyshev’s inequality in order to show that

\[
P(X > 2nH_n) \leq \frac{\pi^2}{6(\ln n)^2}.
\]

Note: You can use Euler’s solution to the Basel problem, the identity \( \sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6} \).

c. Define appropriate events and use the union bound in order to show that

\[
P(X > 2nH_n) \leq \frac{1}{n}.
\]

Note: \( a_n = (1 - \frac{1}{n})^n \) is a strictly increasing sequence with limit \( e^{-1} \).
4. **Coupon Collector Convergence**

In the coupon collector’s problem, there are \( n \) different types of coupons, and you are trying to collect them all. Each time you purchase an item, you receive one of the \( n \) coupons uniformly at random. Let \( T_n \) denote the number of purchases it takes to collect all \( n \) coupons. Prove that \( T_n/(n \ln n) \to 1 \) in probability as \( n \to \infty \).
5. The Weak Law of Large Numbers

Let \( X_1, X_2, \ldots \) be a sequence of i.i.d. random variables with common mean \( \mu \) and MGF \( M_X \). We assume that \( M_X(s) \) is finite when \( s \in (-d, d) \) for some \( d > 0 \). Let

\[
\bar{X}_n := \frac{X_1 + \ldots + X_n}{n}.
\]

a. Show that the transform (or MGF) associated with \( \bar{X}_n \) satisfies

\[
M_{\bar{X}_n}(s) = M_X(s/n)^n.
\]

b. Suppose that the transform \( M_X(s) \) has a first-order Taylor series expansion around \( s = 0 \) of the form

\[
M_X(s) = a + bs + o(s),
\]

where \( o(s) \) is a function that satisfies \( \lim_{s \to 0} o(s)/s = 0 \). Find \( a \) and \( b \) in terms of \( \mu \).

c. Show that for all \( s \in (-d, d) \),

\[
\lim_{n \to \infty} M_{\bar{X}_n}(s) = e^{\mu s}.
\]

*Hint*: If \((a_n)_{n \in \mathbb{N}}\) is a sequence of real numbers converging to \( a \), then \( \lim_{n \to \infty} (1 + \frac{a_n}{n})^n = e^a \).

d. Deduce that \( \bar{X}_n \overset{d}{\to} \mu \). Note that the pointwise convergence of MGFs is equivalent to convergence in distribution.
6. Decoding a Bit from a Noisy Signal

In many wireless communications systems, each receiver listens on a specific frequency. The bit $b$ sent is represented by a $+1$ or $-1$. Unfortunately, noise from other nearby frequencies can affect the receiver’s signal. A simplified model for this noise is as follows: There are $n$ other senders. The $i$th sender is also trying to send a bit $B_i$ represented by $+1$ or $-1$. The receiver obtains the signal $S$ given by

$$S = b + w \sum_{i=1}^{n} B_i,$$

where $w$ is a constant indicating the power of the bits of the other senders.

In order to decode $b$ from $S$, we use the following scheme: if $S$ is closer to $+1$ than $-1$, the receiver assumes that the bit sent was a $+1$; if $S$ is closer to $-1$ than $+1$, the receiver assumes that the bit sent was a $-1$; if $S$ is equidistant to $+1$ and $-1$, the receiver fails to recover $b$.

Assume that all the bits $B_i$ are independent and uniformly distributed over $\{+1, -1\}$.

a. Show that the probability that the receiver cannot determine $b$ correctly is at most $2 \exp\left(-\frac{1}{2nw^2}\right)$. Hint: Transform each $B_i$ in order to use the following Chernoff bound, which you should show is true. If $X$ is Binomial$(n, p)$ as the sum of i.i.d. Bernoulli trials $X_1 + \cdots + X_n$, then

$$\Pr(X \geq nq) \leq \exp\left\{-n \cdot (qt - \ln M_{X_1}(t))\right\}$$

for all $t > 0$, and $\varepsilon > 0$ such that $q = p + \varepsilon < 1$. Then, you may use without proof the fact that for some value of $t$,

$$-n \cdot (qt - \ln M_{X_1}(t)) \leq -2n\varepsilon^2.$$

b. If we want to ensure that the probability to correctly determine $b$ is at least $1 - \delta = 0.999$, what condition do we need to impose on the power of the noise $w$?

c. What would be the condition on the power of the noise $w$ if we have used Chebyshev’s inequality in order to upper bound the error probability?

d. Discuss how the analysis of the error probability in part (a) compares with the analysis of the error probability using Chebyshev’s inequality.