Final Exam

Last Name	First Name	SID

Rules.

- You have 170 minutes (3:10pm - 6:00pm) to complete this exam.
- The maximum you can score is 130.
- The exam is not open book, but you are allowed one side of a sheet of handwritten notes; calculators will be allowed. No phones.
- No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.

Please read the following remarks carefully.

- Show all work to get any partial credit.
- Take into account the points that may be earned for each problem when splitting your time between the problems.

Problem	points earned	out of
Problem 1		40
Problem 2		30
Problem 3		20
Problem 4		20
Problem 5		20
Problem 6		5 Bonus
Total		130 (+5 Bonus)

Problem 1 [40] Short Answers

(a) [5] Alice and Bob play the following game. Alice tosses n identical coins, each of which comes up Heads with probability p. All coins that come up Tails are removed and Alice flips the remaining coins again. If there are X coins that come up Heads this time, Bob gets X dollars. Find the pmf for X. Your answer should contain no summation signs.

(b) [5] The joint distribution of two random variables X and Y is given by: $p_{X,Y}(0,0) = p_{X,Y}(1,0) = 0.2$, $p_{X,Y}(0,1) = p_{X,Y}(1,1) = 0.3$. What is $E[\frac{10}{p_{Y|X}(Y|X)}]$?

(c) [5] The Figure describes $F_X(x)$, the CDF of X. Find $E(X^2)$.

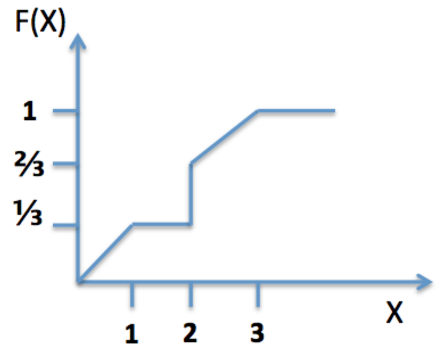

Figure 1: CDF

(d) **[5]** Email to Bob gets automatically classified and those considered to be Spam are routed to a Spam folder. Email not classified as spam is routed to an Inbox. Regular email arrives as a Poisson Process with rate 10 msgs/hour and Spam arrives at a rate of 20 msgs/hour. A regular email is classified as spam with probability 0.05 and a spam email is classified as regular mail with probability 0.01. Bob checks his Inbox after a long time and selects a message at random. What is the probability it is spam?

(e) **[5]** Let (X, Y) be picked uniformly in the unit circle centered at $(0, 0)$. What is $Q[Y|X]$ (the quadratic least squares estimate of Y given X)? *Hint.* Recall that the quadratic least squares estimate of Y given X is the quadratic function of X which minimizes the expected squared distance to Y among all quadratic functions of X: $E[(Y - Q[Y|X])^2] \leq E[(Y - aX^2 - bX - c)^2]$ for all $a, b, c \in \mathbb{R}$.

(f) **[5]** Let X, Z be i.i.d. $\mathcal{N}(0,1)$ and $Y = X + Z$. Find $E[X^2|Y]$.

(g) **[5]** Let $X \in \{0,1\}$ and $Y = (X+1)Z$ where $Z = Expo(1)$ is independent of X. Find $MLE[X|Y=y]$.

(h) **[5]** Let X, Y be i.i.d. $\mathcal{N}(0,1)$. Find $Q[(X+2Y)^2|Y]$.

Problem 2 [30]

Part 1 [10].

The CTMC $\{X_t, t \geq 0\}$ has the transition diagram shown in Figure 2. Let $T_2 = \min\{t \geq 0 \mid X_t = 2\}$. Find $E[T_2 | X_0 = 0]$.

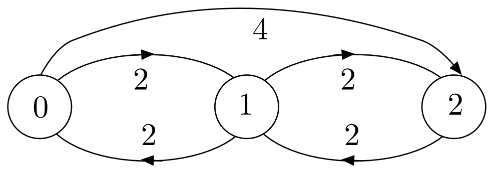

Figure 2: State transition diagram for Problem 2.

Part 2 [20].

Consider the three-state CTMC in Figure 3. The number on the edge directed from state i to state j is $q_{i,j}$, i.e., the transition rate from i to j. Assume that the process is in steady state, i.e., has its invariant distribution.

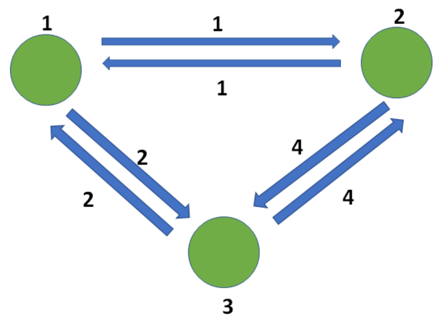

Figure 3: Rate transition diagram for Problem 2-B.

(a) **[5]** Find the long term time average fraction of time spent in state i for each i.

(b) **[5]** Given that the process is in state i at time t, find the average time after time t until the process leaves state i, for each i.

(c) **[5]** Find the long term time average fraction of transitions that go into state i, for each i.

(d) **[5]** Find the steady state probability that the next state to be entered is state 1.

Problem 3 [20]

Part 1 [10].

Let $X = B(p)$. and assume that $X' \in \{0,1\}$ is such that $P[X' = x'|X = x] = P(x, x')$ for $x, x' \in \{0,1\}$. Assume also that Y is such that $P[Y = y|X = x, X' = x'] = Q(x', y)$ for $x, x' \in \{0,1\}$ and $y \in \{1, \ldots, M\}$. Compute $P[X' = 1|Y = y]$. (*Hint:* Bayes' rule.)

Part 2 [10].

Consider a $HMC(\pi, P, Q)$ where P is on $\{0,1\}$. That is, X_n is a Markov chain on $\{0,1\}$ with transition matrix P and $P[Y_n = y|X_n = x] = Q(x, y)$. Let $\hat{X}_n = E[X_n|Y^n]$.

(a) [4] Explain clearly why one should be able to compute $\hat{X}_{n+1}$ as a function of $\hat{X}_n$ and Y_{n+1}. (*Hint:* Try to relate this problem to Part 1.)

(b) [**6**] Derive the equations $\hat{X}_{n+1} = g(\hat{X}_n, Y_{n+1})$.

Problem 4 [20]

Consider the following dynamics equations (all random variables are zero-mean scalars):

$$\begin{aligned} X_0 &\sim \mathcal{N}(0, \sigma_0^2) \\ X_1 &= bX_0 + U, \\ X_2 &= a_0 X_0 + a_1 X_1 + W, \end{aligned}$$

where $U \sim \mathcal{N}(0, \sigma_U^2)$, $W \sim \mathcal{N}(0, \sigma_W^2)$, and (X_0, U, W) are independent.

(a) [**4**] What is the MMSE estimate of X_2 given X_0?

(b) **[4]** What is the MMSE estimate of X_2 given (X_0, X_1)?

(c) **[12]** Now, suppose that X_1 is replaced by an MMSE estimate $Y_1 := X_1 + V$, where $V \sim \mathcal{N}(0, \sigma_V^2)$ is independent of (X_0, U, W). What is the MMSE estimate of X_2 given (X_0, Y_1)?

Problem 5 [20]

Consider two independent random variables $X_1 \sim \mathcal{N}(\mu_1, 1)$, $X_2 \sim \mathcal{N}(\mu_2, 1)$ (where μ_1, μ_2 are unknown), and we would like to detect if $\mu_1 \neq \mu_2$ subject to the constraint that the probability of false alarm is at most α, where $\alpha \in (0, 1)$.

(a) **[6]** You are allowed to observe a linear combination of the two random variables:

$$Y := aX_1 + bX_2.$$

Explain clearly how you should choose a and b.

(b) **[14]** Now suppose the null hypothesis is $\mu_1 = \mu_2$, and the alternate hypothesis is $\mu_1 = \mu_2 + \delta$, where $\delta > 0$ is known. You observe Y as before. Give the Neyman-Pearson decision rule to maximize the probability of correct detection with the constraint on the probability of false alarm.

Problem 6 [5 Bonus]

Thank you for taking the course! We hope you learned a lot and had fun along the way. Please let us know how the course went for you. What did you like and what did you dislike? Do you have any feedback for us?