EECS 126 Probability and Random Processes — UC Berkeley, Fall 2022
Kannan Ramchandran — Dec 16, 2022

Final

Last Name	First Name	SID

Left Neighbor Full Name	**Right Neighbor** Full Name	**Room Number**

Rules.

- **Please bubble in your answers FULLY and write all numerical answers clearly. Answers that are not legible or clearly bubbled in may not get credit.**
- You have 70 minutes to complete the exam. (DSP students with $X\%$ time accommodation should spend $70 \cdot X\%$ time on the exam).
- This exam is not open book. You may reference three double-sided handwritten sheets of paper. No calculator or phones allowed.
- Collaboration with others is strictly prohibited. If you are caught cheating, you will receive a 0 on the final and will face disciplinary consequences.
- Write in your SID on every page to receive 1 point.

Problem	points earned	out of
SID		1
Problem 1		14
Problem 2		14
Problem 3		10
Problem 4		10
Problem 5		14
Problem 6		19
Problem 7		12
Problem 8		14
Problem 9		18
Total		126

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1 Random Cut of Random Graph [14 points]

Recall that a *cut* of a graph G is a subset of vertices $T \subseteq G$, and an edge (i, j) is said to be *across* the cut T if and only if exactly only one of its endpoints i or j belongs to T.

Let $G \sim \mathcal{G}(100, 1/4)$ be an Erdős–Rényi random graph on 100 vertices, in which each edge appears independently with probability 1/4. We construct a *random cut* of G by selecting each vertex of G with probability 1/3. Find the expected number of edges that cross this random cut of the random graph G.

- ◯ 400.
- ◯ 450.
- ◯ 500.
- ◯ 550.
- ◯ 600.
- ◯ None of the above.

2 Vogel im Käfig [14 points]

A bird lives on the integers $\mathbb{Z}$. It starts at 0 at time 0. At each time step, it jumps one step left or right with probability $\frac{1}{2}$ each. In other words, if X_n is its position at time n, then $X_{n+1} = X_n + 1$ w.p. $\frac{1}{2}$ and $X_n - 1$ w.p. $\frac{1}{2}$. If p_n is the probability that the bird is outside of the interval $[-\sqrt{n}, \sqrt{n}]$ at time n, find $\lim_{n\to\infty} p_n$.

Give a **numerical** answer to two decimal places. You may use these following values of $\Phi(\cdot)$, the standard normal CDF: $\Phi(-2) \approx 0.02$, $\Phi(-1) \approx 0.16$, $\Phi(-0.5) \approx 0.31$, $\Phi(0.5) \approx 0.69$, $\Phi(1) \approx 0.84$, $\Phi(2) \approx 0.98$.

3 Poisson Arrivals [10 points]

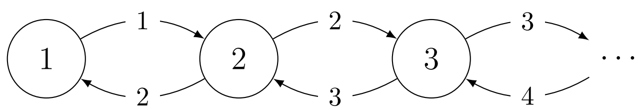
Consider a Poisson process $(N_t)_{t\geq 0}$ with rate $\lambda = 1$. For $i \in Z^+$, let T_i be the time of the ith arrival.

(a) Find $\mathbb{E}[T_3 \mid N(1) = 2]$

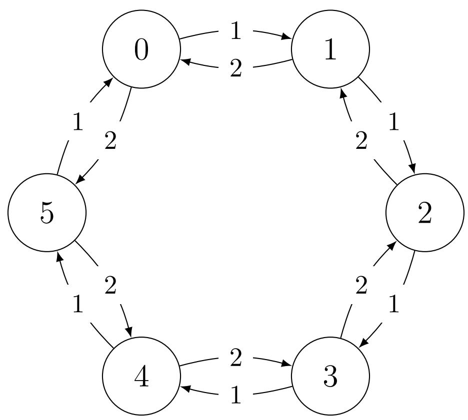
(b) Find $\mathbb{E}[T_2 \mid T_3 = 1]$. Format your answer in reduced fraction form.

4 Reversible CTMCs [10 points]

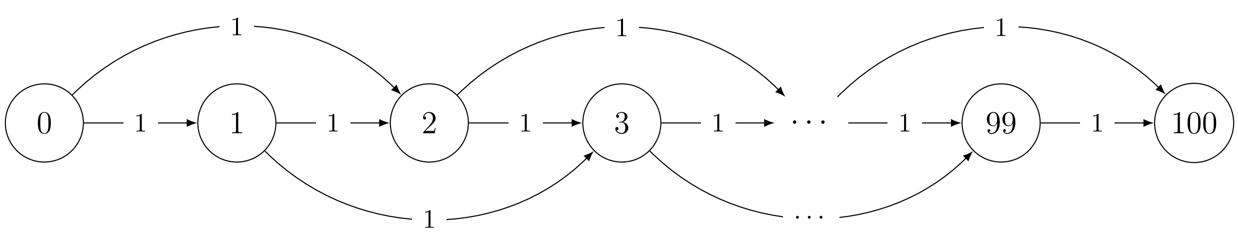
For each of the following transition *rate* diagrams, select true if it describes a *reversible* continuous-time Markov chain and false otherwise.



○ True ○ False



○ True ○ False



○ True ○ False

5 German Tank Problem [14 points]

A bin contains a set of N balls with serial numbers $1, 2, \ldots, N$, where N is unknown. The goal is to estimate N based on a sample of the serial numbers. (Recall that this is the same setting as the German Tank problem discussed in class.) Suppose X is a randomly sampled serial number, i.e. X can take any of the serial numbers from 1 to N with equal probability; if we have a sample of $n \leq N$ serial numbers $X_1, X_2, \ldots, X_n$ sampled at random and without replacement from the bin, and our observed numbers for $n = 4$ are the sequence {33, 7, 100, 44}.

(a) What is the Maximum Likelihood Estimate (MLE) of N given our observed sequence {33, 7, 100, 44} for $n = 4$?

(b) It is possible to show that the expected value of the MLE of N given $n \leq N$ random samples without replacement is given by $\frac{n(N+1)}{n+1}$. Use this to construct an unbiased estimator of N for $n = 4$ given the observed sequence $\{33, 7, 100, 44\}$. (Recall that an unbiased estimator, $\hat{X}$ of a random variable X is such that $\mathbb{E}[\hat{X}]$ is equal to $\mathbb{E}[X]$.)

(c) In a Bayesian setting, suppose the prior distribution on N is Geometric(p) with $p = 0.01$. What is the MAP estimate for N given $n = 4$ and the observed sequence $\{33, 7, 100, 44\}$?

6 Hypothesis Testing [19 points]

Recall the optimization problem solved by the Neyman-Pearson rule:

$$\max_{\hat{X}} \text{PCD} := \Pr\left(\hat{X} = 1 \mid X = 1\right)$$
$$\text{subj. to PFA} := \Pr\left(\hat{X} = 1 \mid X = 0\right) \leq \beta$$

for some fixed $\beta \in [0, 1]$.

(a) Suppose that $Y \mid \{X = 0\}$ and $Y \mid \{X = 1\}$ have the same distribution (e.g. Y is independent of X). Which best describes the relationship between the PFA and PCD for the Neyman-Pearson rule?

- ◯ PFA $\geq$ PCD, but we cannot determine if equality holds without knowing the distribution of Y and/or β.
- ◯ PFA $=$ PCD.
- ◯ PFA $\leq$ PCD, but we cannot determine if equality holds without knowing the distribution of Y and/or β.
- ◯ PFA $= 1 -$ PCD.
- ◯ We cannot determine without further information.

(b) Suppose that $Y \mid \{X = 0\} \sim N(0, 1)$ and $Y \mid \{X = 1\} \sim N(0, 2)$. We solve for the Neyman-Pearson rule with the constraint that our PFA cannot exceed $\beta = 0.3$. Which of the following best describes the shape of the likelihood ratio $L(y)$?

- ◯ monotonically increasing
- ◯ monotonically decreasing
- ◯ increasing and then decreasing
- ◯ decreasing and then increasing
- ◯ none of the above

Now suppose that we have the following conditional distributions for Y:

$$Y \mid \{X=0\} = \begin{cases} 0 & \text{w.p. } 1/6 \\ 1 & \text{w.p. } 1/3 \\ 2 & \text{w.p. } 1/2 \end{cases}$$

$$Y \mid \{X=1\} \sim \text{Uniform}\{0,1,2\}$$

Compute the Neyman-Pearson decision rule $\hat{X}(Y)$ given the constraint that the PFA cannot exceed $\beta = 2/5$. Then, compute the following values. Format your answers as fractions in reduced form.

(c) $\Pr\left(\hat{X} = 1 \mid Y = 0\right)$

(d) $\Pr\left(\hat{X} = 1 \mid Y = 1\right)$

(e) $\Pr\left(\hat{X} = 1 \mid Y = 2\right)$

7 ABC's [12 points]

Let X, Y, and Z be jointly Gaussian random variables with covariance matrix

$$\begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 3 \end{bmatrix}$$

and mean vector $[0, 126, 0]$. We can write $\mathbb{E}[Y|X, Z]$ as $a + bX + cZ$.

Compute a.

Compute b.

Compute c.

8 Hilbert's 25th Problem [14 points]

We will work in the Hilbert space of real-valued random variables $\mathcal{H}$, equipped with the usual inner product $\langle X, Y \rangle = \mathbb{E}(XY)$. Determine whether the following statements are true or false in general.

(a) If X is orthogonal to 1, then X is zero-mean.

◯ True ◯ False

(b) The norm $\|X\| = \sqrt{\langle X, X \rangle}$ always equals the standard deviation $\sigma_X = \sqrt{\text{var}(X)}$.

◯ True ◯ False

(c) X and Y are independent if and only if they are orthogonal.

◯ True ◯ False

(d) Suppose $\text{cov}(X, Y) \neq 0$. Then $\mathsf{proj}_{\{X,Y\}}(Z) \neq \mathsf{proj}_X(Z) + \mathsf{proj}_Y(Z)$.

◯ True ◯ False

9 Estimate the Right Option [18 points]

(a) Which of the following does the expectation of a random variable always minimize (if the expectation exists)?

- ◯ mean squared error, i.e. $\arg\min_{x\in\mathbb{R}} \mathbb{E}\left[(X-x)^2\right] = \mathbb{E}[X]$.
- ◯ mean absolute error, i.e. $\arg\min_{x\in\mathbb{R}} \mathbb{E}\left[|X-x|\right] = \mathbb{E}[X]$.
- ◯ probability of error, i.e. $\arg\min_{x\in\mathbb{R}} \mathbb{P}(X \neq x) = \mathbb{E}[X]$.
- ◯ none of the above

(b) Which of the following is true when we estimate X from Y?

- ◯ The MMSE is always strictly better than the LLSE in terms of mean squared error.
- ◯ If X and Y are both Gaussian, the MMSE equals the LLSE.
- ◯ We can still use the MMSE and the LLSE if the relationship between X and Y is unknown.
- ◯ None of the above.

(c) Which of the following is the estimation error of $\mathbb{L}[X|Y]$ always orthogonal to?

- ◯ all functions of Y
- ◯ all linear functions of Y but not all functions of Y in general
- ◯ all linear functions of X
- ◯ none of the above

(d) Suppose that Y and Z are zero-mean random variables. Decide which of the following statements are true in general. Select **all** correct options.

- ☐ $\mathbb{L}(X \mid Y, Z) = \mathbb{L}(X \mid Y) + \mathbb{L}(X \mid Z)$.
- ☐ If Y and Z are orthogonal, then $\mathbb{L}(X \mid Y, Z) = \mathbb{L}(X \mid Y) + \mathbb{L}(X \mid Z)$.
- ☐ If $X = aY + bZ$, then $\mathbb{L}(X \mid Y, Z) = \mathbb{L}(X \mid Y) + \mathbb{L}(X \mid Z)$.
- ☐ None of the above.

(e) You want to determine the value of $X \sim \mathcal{N}(0,1)$. However, your measurements are imprecise: you observe Y_1 and Y_2, where each Y_i is X plus some independent noise $Z_i \sim \mathcal{N}(0,1)$. Find the MMSE estimate of X given $Y_1 = -4$ and $Y_2 = 10$.

- ◯ 0
- ◯ 1
- ◯ 2
- ◯ 3
- ◯ 6

 Student ID: ______________________________

Use this space for scratch work!

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Use this space for scratch work!