EECS 126 Probability and Random Processes — University of California, Berkeley: Fall 2018
Abhay Parekh and Kannan Ramchandran — September 19, 2018

Midterm 1

Last Name	First Name	SID

- You have 10 minutes to read the exam and 90 minutes to complete this exam.
- The maximum you can score is 120, but 100 points is considered perfect.
- The exam is not open book, but you are allowed to consult the cheat sheet that we provide. No calculators or phones. No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.
- A correct answer without justification will receive little, if any, credit.
- Take into account the points that may be earned for each problem when splitting your time between the problems.

Problem	points earned	out of
Problem 1		40
Problem 2		20
Problem 3		20
Problem 4		20
Problem 5		20
Total		120

Problem 1: Answer these questions briefly but clearly. [40]

(a) Covariance and Independence [5]

Exhibit a pair of random variables (X, Y) such that X and Y are dependent, but $\text{cov}(X, Y) = 0$

Let $X = \text{Uniform}([-4, 4])$ and $Y = X^2$.

Then we have $\mathbb{E}(XY) = \mathbb{E}(Y\mathbb{E}(X|Y)) = 0$. In addition, since $\mathbb{E}(X) = 0$, we have $\text{cov}(X, Y) = 0$.

(b) MGF [5] Let $M_X(s)$ be the moment generating function of a random variable X. Which of the following are valid moment generating functions? If valid, prove which random variable (as a function of X) the MGF belongs to. If invalid, justify.

1. $M_X(s)M_X(2s)$
2. $2M_X(s)$
3. $e^{-2s}M_X(s)$

(i): We know that $M_X(2s)$ is the MGF of $2X$, hence $M_X(s)M_X(2s)$ is the MGF of $X + 2X'$ where X' is an independent copy of X. (ii): we know that MGF evaluated at zero must give 1, however $2M_X(0) = 2 \neq 1$, hence can not be a valid MGF. (iii): $e^{-2s}M_X(s) = \mathbb{E}(e^{s(X-2)})$ and hence is the MGF of $X - 2$.

(c): Book Sale [10] Bob is at a book sale. There are T books in all and L books that he likes. Bob picks up a book at random and buys it if he likes it. Books that are considered once are not considered again. Let X be the number of books he must examine to find n books that he likes. (Here, $n, L, T \in \mathbb{N}$ are fixed numbers ($n \leq L \leq T$), while X is a random variable. The final answers should not involve summations but can have expressions like $\binom{n}{k-1}$, as well as factorials.)

1. Find $E[X]$. (Hint: You do not require the density to find this. Think about symmetry and then use linearity).
2. Find $P(X = x)$. (Hint: Find the right quantity to condition on).

1. Let X_j be the number of non-liked books between the $j-1$st and the jth liked book tested. Then, $X = n + (\Sigma_{j=1}^{n} X_j)$ (noting that we include the n liked books we sample aside from the non-liked ones). By symmetry, the expected number of non-liked books between any two liked books ($\mathbb{E}[X_j]$) is the same and there are $L+1$ such 'slots'. Thus, it follows that $\mathbb{E}[X_j] = \frac{T-L}{L+1}$. Lastly, it then follows that $\mathbb{E}[X] = n\frac{T-L}{L+1} + n = n\frac{T+1}{L+1}$.

2. Consider the event A_x that the x^{th} book tested is liked. By symmetry in sampling without replacement, the probability that this happens is $\frac{L}{T}$. Now, conditioned on A_x, the probability that $X = x$ corresponds to exactly $n-1$ likeable books in the first $x-1$ books and $x-n$ non-liked books in the first $x-1$ books. The total number of ways for this to happen is $\binom{L-1}{n-1}\binom{T-L}{x-n}$. The total number of ways to pick the $x-1$ books at the front is $\binom{T-1}{x-1}$. Finally, notice that any sequence of books is equally likely. Then,

$$P(X = x) = \frac{\binom{L-1}{n-1}\binom{T-L}{x-n}}{\binom{T-1}{x-1}} \frac{L}{T}$$

.

(d) Points in square [5] Two points are placed uniformly at random and independently in $[0,1]^2$. What is the expected value of the square of the distance between the two points?

Let (X_1, Y_1) and (X_2, Y_2) be the two points. The quantity of interest is $D = (X_1 - X_2)^2 + (Y_1 - Y_2)^2$. Note that X_1, X_2 are independent and uniformly distributed in $[0,1]$, and so are Y_1, Y_2. Hence, $\mathbb{E}((X_1 - X_2)^2) = 2\mathbb{E}(X_1^2) - 2\mathbb{E}(X_1)^2 = 2\text{Var}(X_1) = 1/6$. Similarly, $\mathbb{E}((Y_1 - Y_2)^2) = 1/6$. Therefore, $\mathbb{E}(D) = 1/3$.

(e): Light bulbs [5] A lighting company tests 10 bulbs by turning them on at the same time. Each bulb has a lifetime exponentially distributed with parameter λ. Let X be the length of the time interval between the time that the first bulb burns out and the second bulb burns out. Find $f_X(x)$.

First, notice that the distribution of $X_{(2)} - X_{(1)}$ is independent of $X_{(1)}$. This can be justified by the fact that conditioning on a particular value of $X_{(1)}$, the additional remaining lifetimes of the $n-1$ bulbs is independent of $X_{(1)}$ by memorylessness of the exponential distribution. Then, we can view $X_{(2)} - X_{(1)}$ as the time till the death of the first of the remaining $n-1$ bulbs, and can hence conclude that it is distributed as the minimum of $n-1$ i.i.d. exponential random variables. Finally, using the formula for ordered statistics derived in discussion, we conclude that $X_{(2)} - X_{(1)}$ is itself exponentially distributed with rate $(n-1)\lambda$, hence $f_{X_{(2)}-X_{(1)}}(x) = (n-1)\lambda e^{-(n-1)\lambda x}$.

(f) Fountain Codes [10]

100 data chunks need to be transmitted over a packet erasure channel (as in Lab 2). We will encode the data chunks into 200 packets using the following scheme. For each packet, we will first roll a 6-sided fair die, and based on the outcome, sample uniformly at random without replacement that many of the 100 data chunks and XOR the data chunks. (As an example, if the die comes up "3", then the corresponding packet will be the XOR of three uniformly sampled random data chunks from 1 to 100, say data chunks $4, 43$ and 87).

1. What is the probability that a data chunk is connected to a given packet?
2. What is the average number of packets that a data chunk will be associated with?
3. What is the probability that a data chunk is not connected to any packet?

1. We compute the probability that a chunk X_i is connected to some packet Y_j. Suppose the outcome of the die is d. Conditioned on this,

$$P(x_i \in Y_j | D_j = d) = d/100$$

$$P(x_i \in Y_j) = \sum_d P(x_i \in Y_j | D_j = d) p_D(d) = \sum_d \frac{d \cdot p_D(d)}{100} = 3.5/100$$

2. Since a chunk is connected to each packet with probability 3.5/100, the expected number of packets a chunk is part of is $200 \times 3.5/100 = 7$ by linearity of expectation.

3. The probability that a data chunk is not connected to any given packet is $(1 - \frac{3.5}{100})$. So the required probability is $(1 - \frac{3.5}{100})^{200}$.

Problem 2: Magnets [20]

There are n bar magnets, $n > 1$, placed in a line end to end. Assume that each magnet takes one of the two possible orientations, say (NS) or (SN), with equal probability, and magnets have independent orientations. Adjacent magnets with like poles repel, while those with opposite poles join and form blocks. For instance, if $n = 5$, and the orientation of magnets is $(NS)(SN)(SN)(NS)(NS)$, they form 3 blocks of the form $(NS) \quad | \quad (SN)(SN) \quad | \quad (NS)(NS)$. Let N be the number of blocks of joint magnets.

1. What is $\mathbb{E}(N)$?

   Let X_i, $1 \leq i \leq n$, be the orientation of the ith magnet. Also, let I_i, $1 \leq i \leq n-1$, be the indicator of the event that magnets i and $i+1$ have opposite orientation. Indeed, $N = 1 + \sum_{i=1}^{n-1} I_i$. Moreover, $\mathbb{E}(I_i) = 1/2$ for $1 \leq i \leq n-2$. Therefore, $\mathbb{E}(N) = 1 + (n-1)/2 = (n+1)/2$.

2. What is $\mathrm{Var}(N)$?

   With the above notation, we have $\mathbb{E}(I_i I_j) = 1/4$ for $i \neq j$. Therefore,

$$\mathrm{Var}(N) = \mathrm{Var}(N-1) = \mathbb{E}\left(\left(\sum_{j=1}^{n-1} I_j\right)^2\right) - \left(\frac{n-1}{2}\right)^2 = (n-1)/4.$$

Problem 3: Squared sum of Gaussians [20]

Let X and Y be independent Gaussian random variables with mean 0 and standard deviation 1.

1. Derive the probability density function (pdf) of $X^2 + Y^2$.

   Consider the MGF of X^2

   $$M_{X^2}(s) = \mathbb{E}[e^{sX^2}] = \int_{-\infty}^{\infty} e^{sx^2} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(1/2-s)x^2} dx$$

   Notice the integral on the right resembles that of a Gaussian pdf with $\frac{1}{2\sigma^2} = 1/2 - s \implies \sigma^2 = \frac{1/2}{1/2-s}$. So the value of the integral is just $1/(normalization\ constant) = \sqrt{2\pi\sigma^2} = \sqrt{2\pi\frac{1/2}{1/2-s}}$. So the MGF is

   $$\sqrt{\frac{2\pi\frac{1/2}{1/2-s}}{2\pi}} = \sqrt{\frac{1/2}{1/2-s}}$$

   Since X^2 and Y^2 are independent, we know

   $$M_{X^2+Y^2}(s) = M_{X^2}(s)M_{Y^2}(s) = \frac{1/2}{1/2-s}$$

   We see this is the MGF of an exponential with parameter $1/2$. So the pdf of $X^2 + Y^2$ is $\frac{1}{2}e^{-z/2}$.

   Alternative solution: Start with the cdf.

   $$P(X^2 + Y^2 < z) = \iint_C f_{X,Y}(x, y)dxdy$$

   $$= \iint_C \frac{e^{-(x^2+y^2)/2}}{2\pi} dxdy = \frac{1}{2\pi} \iint_C e^{-r^2/2} r dr d\theta = \int_0^{z/2} e^{-u} du = 1 - e^{-z/2}$$

   where C is the circle of radius $\sqrt{z}$. The pdf is therefore

   $$\frac{d(1 - e^{-z/2})}{dz} = \frac{1}{2}e^{-z/2}$$

   $Z = X^2 + Y^2$ is therefore distributed $Exp(1/2)$.

2. For $t > 2$, provide upper bounds on $\mathbb{P}(X^2 + Y^2 > t)$ using Markov and Chebyshev inequality.

   By Markov's inequality with $Z = X^2 + Y^2$,

   $$P[Z > t] \leq \frac{E[X^2 + Y^2]}{t} = 2/t$$

From Cheybyshev's, we have

$$P[Z > t] = P[Z - 2 > t - 2] \leq P[|Z - 2| > t - 2] \leq 4/(t - 2)^2$$

Problem 4: Graphical Density [20]

Let (X, Y) be uniformly distributed over the triangle with vertices $(0, 0)$, $(1, 0)$, and $(2, 1)$.

1. Find $f_{X,Y}(x, y)$ and $f_X(x)$.
2. Compute $\mathbb{E}[Y \mid X = x]$.

1. Since the density is uniform, we need only calculate the total area of the triangle, which is $(1 \cdot 1)/2 = 1/2$. Therefore, the density is

$$f_{X,Y}(x, y) = 2, \qquad (x, y) \in T,$$

where T is the triangle.

We integrate out y:

$$\begin{aligned} f_X(x) &= \mathbb{1}\{0 \le x \le 1\} \int_0^{x/2} 2 \, \mathrm{d}y + \mathbb{1}\{1 \le x \le 2\} \int_{x-1}^{x/2} 2 \, \mathrm{d}y \\ &= x \cdot \mathbb{1}\{0 \le x \le 1\} + (2 - x) \cdot \mathbb{1}\{1 \le x \le 2\}. \end{aligned}$$

The density can equivalently be written as

$$f_X(x) = (1 - |x - 1|)^+,$$

where $x^+ = \max(0, x)$.

2. If $0 \le x \le 1$, then $Y \mid X = x$ is uniform on a slice from 0 to $x/2$, so $E[Y \mid X = x] = x/4$. Similarly, if $1 \le x \le 2$, then $Y \mid X = x$ is uniform on a slice from $x - 1$ to $x/2$, so $E[Y \mid X = x] = (3x - 2)/4$.

Problem 5: Sum of Poisson Squares [20] Let $X_i, i \geq 1$ be i.i.d. Poisson random variables with parameter λ. Also, let N be a geometric random variable with parameter p which is independent from all X_i's. With this, define $S := X_1^2 + X_2^2 + \cdots + X_N^2$.

1. Find $\mathbb{E}(S)$.

   Using the law of iterated expectations, we have

   $$\mathbb{E}(S) = \mathbb{E}(\mathbb{E}(S|N)) = \sum_{n=1}^{\infty} \mathbb{E}(S|N=n) \Pr(N=n).$$

   Since N is independent from X_i's, we have $\mathbb{E}(S|N=n) = n\mathbb{E}\left(X_1^2\right) = n(\mathrm{Var}(X_1) + \mathbb{E}(X_1)^2) = n(\lambda + \lambda^2)$. Hence, $\mathbb{E}(S|N) = N(\lambda + \lambda^2)$ and

   $$\mathbb{E}(S) = (\lambda + \lambda^2)\mathbb{E}(N) = \frac{\lambda + \lambda^2}{p}.$$

2. For an integer $k \geq 1$, find $\mathbb{E}(S|N > k)$.

   Again, using the law of iterated expectations, we have

   $$\begin{aligned} \mathbb{E}(S|N>k) = \mathbb{E}(\mathbb{E}(S|N)\,|N>k) &= \sum_{n=k+1}^{\infty} \mathbb{E}(S|N=n, N>k) \Pr(N=n|N>k) \\ &= \sum_{n=k+1}^{\infty} \mathbb{E}(S|N=n) \Pr(N=n-k) \end{aligned}$$

   where in the second line, we have used the memoryless property of the geometric distribution to substitute $\Pr(N=n|N>k)$ with $\Pr(N=n-k)$ for $n>k$. Using the calculations in the previous part, we have

   $$\begin{aligned} \mathbb{E}(S|N>k) &= \sum_{n=k+1}^{\infty} (\lambda+\lambda^2)n\Pr(N=n-k) \\ &= (\lambda+\lambda^2)\sum_{n=1}^{\infty}(n+k)\Pr(N=n) \\ &= (\lambda+\lambda^2)\mathbb{E}(N+k) \\ &= (\lambda+\lambda^2)\left(\frac{1}{p}+k\right). \end{aligned}$$