EECS 126 Probability and Random Processes — UC Berkeley, Spring 2020
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Midterm 1

Last Name	First Name	SID

Left Neighbor First and Last Name	**Right Neighbor** First and Last Name

Rules.

- **Unless otherwise stated, all your answers need to be justified.**
- You have 10 minutes to read the exam and 90 minutes to complete it.
- The exam is not open book. No calculators or phones allowed.
- Write in your SID on every page to receive 1 point.

Problem	points earned	out of
Problem 1		62
Problem 2		12
Problem 3		12
Problem 4		18
SID		1
Total		105

1 Assorted Problems (62 points)

(a) **True / False (6 points)**

Suppose $X \sim \text{Exponential}(\lambda)$ and $Y \sim \text{Exponential}(\mu)$ are exponentially distributed random variables and are independent. Then they are uncorrelated. Justify your answer.

◯ True ◯ False

Suppose again $X \sim \text{Exponential}(\lambda)$ and $Y \sim \text{Exponential}(\mu)$ are exponentially distributed random variables and are independent. Then their sum is exponentially distributed. Assume, $\lambda > \mu$, and justify your answer.

◯ True ◯ False

(b) **Drawing Balls (6 points)**

Alan and 3 of his friends are drawing balls out of a bag. In the bag, there are 3 red balls, 4 white balls, and 5 blue balls. After his 3 friends draw without replacement, Alan takes his turn. What's the probability that he draws a blue ball?

(c) **Covariance (6 points)**

Suppose $X, Y \sim \text{Uniform}[-\frac{1}{2}, \frac{1}{2}]$, and $\text{var}(X - Y) = 1$. What is $\text{cov}(X, Y)$?

(d) **Exponential Race (6 points)**

Alice and Bob are serving customers at a store counter. Their service times are modeled as independent exponential random variables, with Alice's rate being twice that of Bob's. Suppose you arrive at the counter to find that Alice and Bob are both busy with customers, and there are no other customers in the store. You will be served as soon as either Alice or Bob is free. What is the probability that you will be the last customer to leave the store?

(e) **Fountain Codes (6 points)**

You have 10 chunks to decode using fountain codes. You have three 2-degree packets, five 3-degree packets, one 4-degree packet and one 1-degree packet. All the chunks in the packets are drawn uniformly at random. What is the expected number of 3-degree packets after peeling the 1-degree packet off?

As an example, if a 3-degree packet contains the chunk in the 1-degree packet, after peeling the 1-degree packet off, it will become a 2-degree packet.

E[# 3-degree packets] =

(f) **Throwing Axe (6 points)**

You go axe-throwing with your friends and you noticed an axe sticking out of the board from the previous session. $\frac{1}{4}$ of axe throwers are left-handed. and $\frac{3}{4}$ are right-handed. Let X be a continuous random variable representing where an axe lands on the horizontal axis relative to the middle. You know that left-handed throwers land their axe according to a $X \sim N(-1, 1)$ distribution. Right-handed throwers land their axe according to a $X \sim N(1, 1)$ distribution. Given that you see an axe sticking out of the board at $X = \frac{-1}{2}$, what is the probability that the person who threw the axe was a right-handed thrower?

(g) **Telephone (6 points)**

Let $X_1 \sim \mathcal{N}(0,1)$ and $X_i \sim \mathcal{N}(X_{i-1},1)$. What is the distribution of X_n?

(h) **Cutting the Rope (6 points)**

Consider a rope with length 1. One uniformly and independently picks two places on the rope and cuts the rope to three pieces. Denote Z as the length of the middle piece. Find the CDF of Z and $\mathrm{E}(Z)$.

$F_Z(z) =$

$\mathrm{E}[Z] =$

(i) **MGF for Binomial (6 points)**

For $Y \sim \text{Binom}(n, 1/2)$, show that the MGF of $Y - \text{E}[Y]$ is at most $e^{ns^2/8}$.
Hint: You can use the fact that $\frac{1}{2}(e^{s/2} + e^{-s/2}) \leq e^{s^2/8}$.

(j) **Tight Inequalities (8 points)**

(i) (4 points) The Markov bound is generally quite loose, but for certain random variables and certain values it can be tight. For a fixed k, construct a random variable X that assumes only non-negative integer values such that $P(X \geq k) = \frac{\text{E}[X]}{k}$.

(ii) (4 points) For a fixed k, construct a random variable X that is tight for the Chebyshev inequality, or show that this is impossible. Justify your answer.

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2 Good and Bad Items (12 points)

Suppose you have N items, G of which are good and B of which are bad. You start to draw items without replacement, and suppose that the first good item appears on draw X.

(a) (6 points) What is $\mathrm{E}[X]$?

(b) (6 points) What is $\mathrm{var}(X)$? No need to simplify your answer.

3 Urns and Balls (12 points)

Two urns contain a large number of balls with each ball marked with one number from the set $\{0, 2, 4\}$. The proportion of each type of ball in each urn is displayed in the table below:

		Number on Ball X		
		0	2	4
Urn Label θ	A	0.6	0.3	0.1
	B	0.1	0.3	0.6

An urn is randomly selected with equal probability and then a ball is drawn at random from the urn. The urn from which the ball is selected (either A or B) is represented by θ and the number on the ball is represented by X.

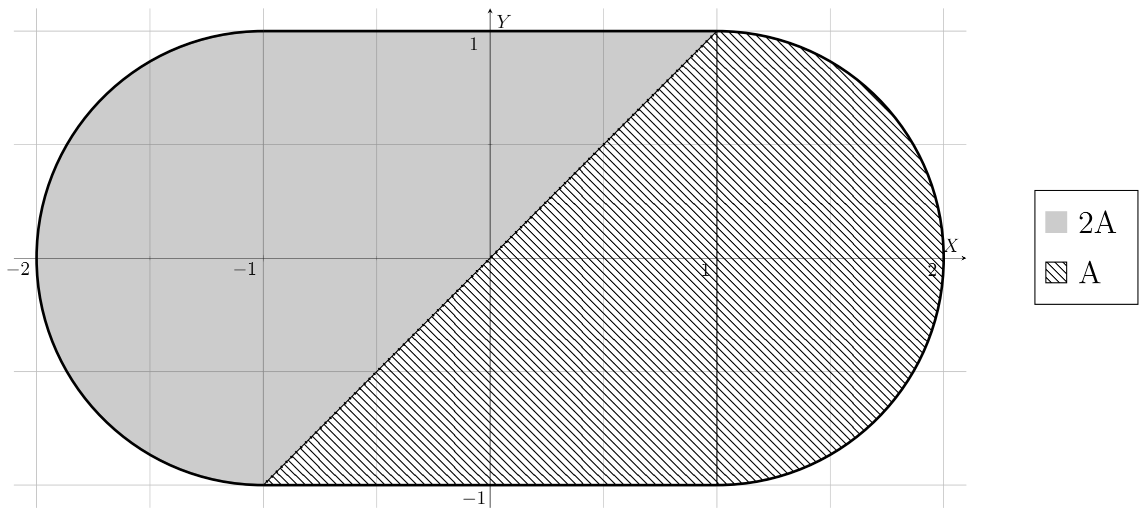
(a) (5 points) Find $\text{var}(\text{E}[X|\theta])$ (place this value in box).

(b) (5 points) Find $\text{E}[\text{var}(X|\theta)]$ (place this value in box).

(c) (2 points) Calculate $\text{var}(X)$.

4 Graphical Density (18 points)

Random variables X and Y have a joint density as pictured below:



(a) (5 points) What is A?

(b) (5 points) In terms of A, what is the marginal density of X?
Hint: The equation of a circle of radius 1 centered at (a, b) is $(x-a)^2 + (y-b)^2 = 1$.

$$f_X(x) = \begin{cases} & \text{if } x \in [-2,-1] \\ & \text{if } x \in [-1,1] \\ & \text{if } x \in [1,2] \end{cases}$$

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(c) (5 points) Suppose $\mathrm{E}[X \mid X \in [1,2]] = 1 + \frac{4}{3\pi}$. Let Z be an indicator if (X, Y) is in the region with density A. What is $\mathrm{E}[X \mid Z = 1]$? You don't need to simplify your answer.

(d) (3 points) Suppose the answer to the previous question is a. In terms of that, what is $\mathrm{E}[X]$?