EECS 126 Probability and Random Processes — UC Berkeley, Spring 2022
Kannan Ramchandran — February 22, 2022

Midterm 1

Last Name	First Name	SID

Left Neighbor First and Last Name	**Right Neighbor** First and Last Name

***Rules*.**

- **Unless otherwise stated, all your answers need to be justified and your work must be shown. Answers without sufficient justification will get no credit.**
- You have 80 minutes to complete the exam. (DSP students with $X\%$ time accommodation should spend $80 \cdot X\%$ time on the exam).
- This exam is not open book. You may reference one double-sided handwritten sheet of paper. No calculator or phones allowed.
- Collaboration with others is strictly prohibited. If you are caught cheating, you may fail the course and face disciplinary consequences.
- Write in your SID on every page to receive 1 point.

Problem	points earned	out of
SID		1
Problem 1		49
Problem 2		14
Problem 3		26
Problem 4		21
Total		111

1 Potpourri [7 + 7 + 7 + 7 + 7 + 7 + 7 points]

(a) **Foraging [7 points]**
You find 2 different plants. A plant is green with probability $\frac{2}{3}$ and edible with probability $\frac{2}{5}$. Each plant's properties are independent of each other, and a plant's color is independent of its edibility. You are told that at least one of them is both edible and green. What is the probability that both plants are edible?

(b) **Comparing Gaussians [7 points]**
Let $X \sim \mathcal{N}(0, 1)$ and $Y \sim \mathcal{N}(2, 3)$, where X and Y are independent. What is $\mathrm{P}(X > Y)$? You may leave your answer in terms of Φ, the CDF of the standard normal distribution.

(c) **Uncorrelatedness [7 points]**
If X and Y are uncorrelated, are X^2 and Y also uncorrelated? Provide complete justification for full credit. Additionally, provide a counterexample if the answer is false.

(d) **Important Moments [7 points]**
Given that $\mathrm{E}[X^i] = i!\beta^i$ for $i = 0, 1, 2, \ldots$ what is the distribution of X? Provide complete justification for full credit.

(e) **Exponential Sampling [7 points]**
Suppose you are able to generate values from $U \sim \text{Uniform}[0, 2]$. How can you simulate and sample values from an exponential distribution using U and $F_X(x)$, where $X \sim \text{Exponential}(\lambda)$? Provide complete justification for full credit.

(f) **Matrix Sketching [7 points]**

In lab, we studied Gaussian Sketch and Count Sketch; let's consider the following sketching matrix: *Random Sign Sketch*. Suppose that the entries of $S \in \mathbb{R}^{d \times n}$ are generated such that

$$S_{ij} \sim_{\text{iid}} \begin{cases} +1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases}$$

Compute $\mathrm{E}[(S^\top S)_{ij}]$ and $\mathrm{var}[(S^\top S)_{ij}]$. Your answer may depend on d and/or n.

(g) **Polling for IQ [7 points]**
You and Clark discover a new planet called `621 SCEE`! You poll 1 million creatures on this planet for their IQ score (can be negative), but accidentally lose everyone's individual scores. However, you remember that the sample mean is $\frac{1}{2}$ and the sample variance is $\frac{3}{16}$.

With no further assumptions on the scores' possible distribution, choose from Markov's and Chebyshev's inequalities. What is the tightest upper bound on the number of scores ≥ 1?

2 Minecraft! [7 + 7 points]

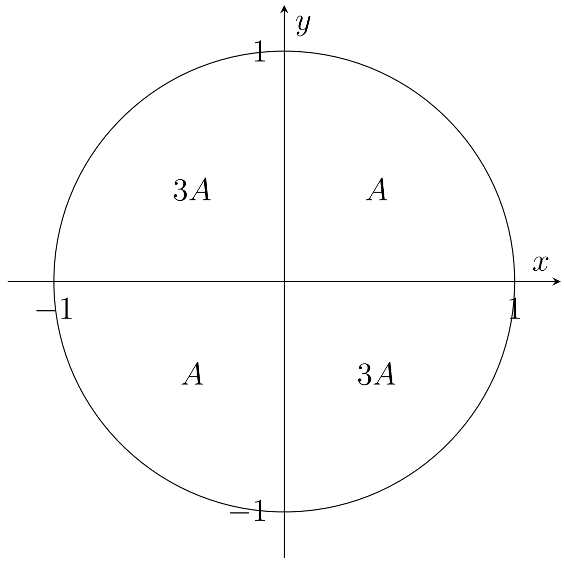
Merrick is mining for diamonds in Minecraft! His pickaxe has rating $R \sim \text{Uniform}\{1, 2, 3, 4, 5\}$ and mining speed $S \sim \mathcal{N}(R, R/2)$ (note that it is possible to have negative mining speed). After a certain time T, Merrick stops mining. For any pickaxe speed S, $\mathrm{P}(T \leq t) = 1 - e^{-t/S^2}$.

(a) What is $\mathrm{var}(S)$?

(b) What is $\mathrm{E}[T]$?

3 Graphical Density [4 + 6 + 7 + 9 points]

Let X and Y have joint probability density function as below.



Note that the circle of radius 1 is described by the equation $x^2 + y^2 = 1$.

(a) Find the value of A.

(b) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

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(c) Are X and Y independent?

(d) Find $\text{cov}(X, Y)$. Are X and Y correlated?

4 Some Ticking Tocks [5 + 7 + 9 points]

Axel has a collection of 101 faulty stopwatches, and starts them all of them at once. Suppose that the amount of time each stopwatch runs before stopping by itself, in minutes, is i.i.d. Exponential(2).

(a) What is the probability that each stopwatch lasts at least half a minute?

(b) Find the PDF of Y, the time it takes until all stopwatches stop.

(c) Find the expected value of the median time it takes for the stopwatches to stop. Feel free to express your answer in terms of $H_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$, the nth harmonic number.