EECS 126 Probability and Random Processes — UC Berkeley, Fall 2019
Shyam Parekh — November 5, 2019

Midterm 2

Last Name	First Name	SID

***Rules*.**

- No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.
- You have 10 minutes to read the exam and 90 minutes to complete it.
- The exam is not open book. No calculators or phones allowed.
- Unless otherwise stated, all your answers need to be justified.

Problem	points earned	out of
Problem 1		36
Problem 2		20
Problem 3		30
Problem 4		26
Problem 5		14
Total		126

1 Assorted Problems [36]

(a) **Bipartite Markov Chain** [4]

Consider the following undirected bipartite Markov chain. At each time step, you pick one of your neighbors uniformly at random to transition to. If we start with an arbitary distribution π_0, then the distribution of this Markov Chain after k transitions will always converge to the stationary distribution as $k \to \infty$. Justify your answer.

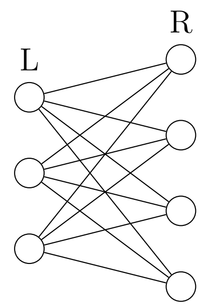


Figure 1: *

Complete Bipartite Graph for $m = 3, n = 4$

○ True ○ False

(b) **School Cancellations** [8]

In Berkeley, power outages happen according to a Poisson Process with a rate of λ_p and independently earthquakes happen according to a Poisson Process with a rate of λ_e. Marc Fisher cancels school with probability p_p if there is a power outage, and p_e if there is an earthquake. What is the expectation and the variance of the amount of time T between the previous school cancellation and the next school cancellation from today? You may assume that this trend has been going on since infinitely in the past.

$E(T) =$

$\text{var}(T) =$

(c) **Entropy of a Markov Chain [8]**

Consider a random walk along the integers (..., -3, -2, -1, 0, 1, 2, 3, ...). You start at 0 at time 0 and pick a direction (positive or negative with equal probability) to move in for time 1. At every time step after 1, you reverse direction with probability 0.25 and take a step in the new direction, or continue in the same direction otherwise. What is $H(X_0, \ldots, X_{n-1})$? You may leave your answer in terms of $H_b(p) = -p \log p - (1 - p) \log(1 - p)$.

(d) **Poisson Process and Covariance [8]**

Consider a Poisson Process $\{N(s) : s \in [0, \infty)\}$ with rate λ.

Find the covariance of $N(t_1)$ and $N(t_2)$ for $t_2 > t_1 \geq 0$.

(e) **Metropolis Hastings [8]**

Answer the following True/False questions about the MCMC lab. Briefly justify your answer **in one or two sentences**.

Metropolis Hastings allows us to generate random samples from a distribution $p(x)$ even if it's intractable to compute.

◯ True ◯ False

Any Metropolis Hastings Markov chain can be made aperiodic.

◯ True ◯ False

Burn in time is how long the Markov chain takes to converge to the state whose stationary distribution probability is maximal.

◯ True ◯ False

In the Traveling Salesman Problem, you want to find the shortest path to visit n cities. Even though this problem has $n!$ possible answers and is NP-hard, we can approximately solve it with Metropolis Hastings. Our plan is to design a Markov chain such that if x^* is the best path, the stationary distribution probability $\pi(x^*)$ will be maximal. Then by running the Markov chain for a long time and looking at the most commonly visited state, we can infer the best path. If $L(x)$ is the length of a path x, then for this problem we could let our directly proportional estimate $f(x)$ be equal to $L(x)$.

◯ True ◯ False

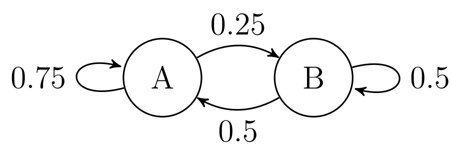
2 Convergence [20]

(a) **Convergent Balls [10]**

Kevin has a basket of k balls. Each ball is either white or red. Initially, all of the balls in his basket is red. Let $X_i^{(n)}$ denote the if the i-th ball is red at time step n. And let $Y_n = \sum_{i=1}^{k} X_i^{(n)}$ therefore be the number of balls in Kevin's basket that are red at time step n. At every time step, Kevin takes a ball out uniformly at random, and he replaces the ball with a white ball. Prove that Y_n converges in probability to 0.

(b) **Markov Chain Groups [10]**

Consider the following Markov Chain



Given a finite length sequence $x_1, \ldots, x_n$ sampled from this Markov chain, we can form groups for samples that are in the same state. For example, if our sequence is

$$\underbrace{A, A, A}_{1}, \underbrace{B, B}_{2}, \underbrace{A, A}_{3}, \underbrace{B, B, B, B, B}_{4}$$

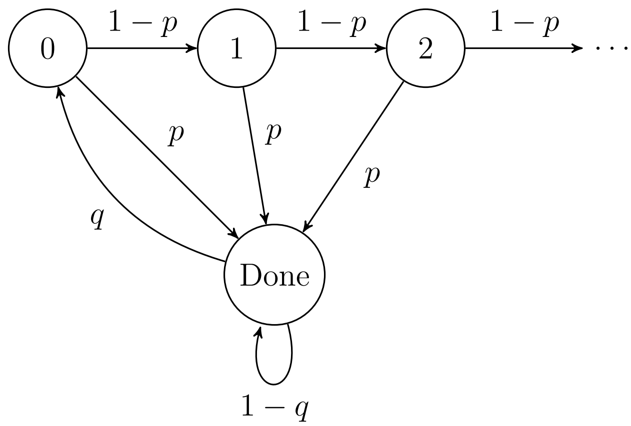
Then we have 4 groups of size 3, 2, 2, 5. Let G_n be

$$G_n = \frac{n}{\text{number of groups among } x_1, \ldots, x_n}$$

In the example, $G_{3+2+2+5} = G_{12} = \frac{12}{4} = 3$. What does G_n converge to?

3 Discrete Time Markov Chains [30]

Consider the following Discrete Time Markov Chain.



(a) **Stationary Distribution [10]**

Find the stationary distribution of the chain.

(b) **Hitting Time Backwards [10]**

What is the expected hitting time from state n to state 0?

(c) **Hitting Time Forward [10]**

Suppose $p = q = \frac{1}{2}$. What is the expected hitting time from state 0 to state n?

Hint: Use parts (a) and (b).

4 Last Arrival [26]

Suppose we have a two independent Poisson Processes X and Y with arrival rates λ and 2λ, respectively.

(a) **X or Y? [6]**

What is the probability that the last arrival for Y comes after the last arrival for X in some given time interval $(0, t)$?

(b) **Expected Arrivals Afterwards [10]**

Suppose we are given that $N_Y(t) = n$. What is the expected number of arrivals N from X after the last arrival from Y on $(0, t)$?

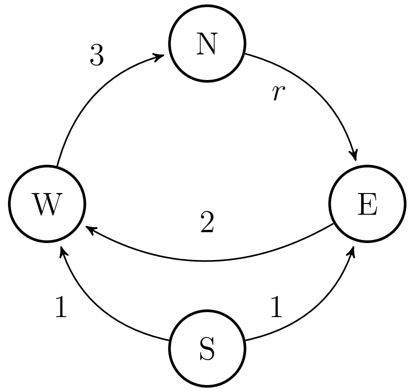
(c) **Chernoff Bound [10]**

Provide an upper bound using the Chernoff bound on the probability that the total number of arrivals from X and Y in the interval $(0, t)$ exceed $k\lambda t$ for some positive integer k. Express your answer in terms of k and simplify your answer as much as possible.

Hint: The MGF for a Poisson random variable X with parameter λ is $M_X(s) = \exp(\lambda(e^s - 1))$.

5 Continuous Time Markov Chains [14]

Consider the following CTMC for $r > 0$.



(a) **Stationary Distribution [7]**

For the above CTMC, find the stationary distribution in terms of r, where $r > 0$.

(b) **Equivalent DTMC [7]**

Draw a DTMC with the same state space with the same stationary distribtion as the above CTMC. If necessary, draw multiple DTMCs depending on the value of r.