EECS 126 Probability and Random Processes — UC Berkeley, Fall 2021
Shyam Parekh — November 4, 2021

Midterm 2

Last Name	First Name	SID

Left Neighbor First and Last Name	**Right Neighbor** First and Last Name

Rules.

- **Unless otherwise stated, all your answers need to be justified and your work must be shown. Answers without sufficient justification will get no credit.**
- You have 80 minutes to complete the exam. (DSP students with $X\%$ time accommodation should spend $80 \cdot X\%$ time on the exam and 10 minutes to submit).
- This exam is not open book. No calculator or phones allowed.
- Collaboration is prohibited. If caught cheating, you may fail and face disciplinary actions.
- Write in your SID on every page to receive 1 point.

Problem	points earned	out of
SID		1
Problem 1		12
Problem 2		11
Problem 3		9
Problem 4		14
Problem 5		8
Problem 6		13
Total		68

1 A Poisson Chain [2 + 2 + 2 + 3 + 3 points]

Andy runs a discrete-time Markov chain $(X_n)_{n\geq 0}$, starting from $X_0 = 0$. At each time step, if the current state is 0, he samples from a Poisson distribution with a fixed parameter $\lambda \in (0, \infty)$, independent of all other samples, and sets the state to the outcome. Otherwise, if the state is non-zero, he decrements the state by 1. In other words, if $Y \sim \text{Poisson}(\lambda)$, the transition matrix P contains the following values:

$$P_{ij} = \begin{cases} \Pr\{Y = j\} & i = 0 \text{ and } j \geq 0 \\ 1 & i > 0 \text{ and } j = i - 1 \\ 0 & \text{otherwise} \end{cases}$$

For each question, please provide a brief justification for full credit.

a) What is the state space of this Markov chain?

b) Is this Markov chain irreducible?

c) What is the period of this Markov chain?

d) Is this Markov chain positive recurrent, null recurrent, or transient? Also, find the expected return time to state 0.

e) Does a unique stationary distribution π exist for this Markov chain? If it does, what is π_0?

2 Coin Flipping! [5 + 6 points]

a) One day, Han is bored and decides to flip coins to pass time. These coins are *special*: they will stand up on the thin edge w.p. (with probability) p, land heads w.p. $9p$, and land tails otherwise. Han flips 10 of these coins and observes 5 heads, 3 tails, and 2 coins standing on their edge. Using MLE, what's the most likely value for p?

b) Now, suppose Clark flips a different coin. This coin is fair (shows only heads and tails with equal probability) with probability $\frac{1}{3}$ and otherwise it is a special coin (as defined in the previous problem, now with $p = 0.05$). Given that Clark flips this coin 10 times and observes h heads and $10 - h$ tails, what is the MAP rule for whether or not he is flipping a special coin? Simplify the MAP rule into a single inequality involving h.

3 Convergence [9 points]

The Central Limit Theorem says that for $(X_i)_{i=1}^{\infty}$ that are i.i.d. mean zero and variance 1,

$$Z_n := \frac{X_1 + \cdots + X_n}{\sqrt{n}} \xrightarrow{\mathrm{d}} \mathcal{N}(0,1).$$

Show that this limit cannot be upgraded to convergence almost surely. *Hint:* It may help to consider the sequence of random variables $(\sqrt{2}Z_{2n} - Z_n)$.

4 Café 126 [4 + 4 + 6 points]

The EECS 126 staff are opening a café! They need your help in planning their business. Suppose that customers arrive according to a Poisson process with rate λ per hour.

a) Let $N(s,t) := N_s - N_t$ be the number of customers that arrive between times t and s. What are $\mathbb{E}[N(3,1) \mid N(6,2)]$ and $\text{Var}[N(3,1) \mid N(6,2)]$?

b) Each customer independently orders exactly one drink, which is a cappuccino with probability 1/2, an espresso with probability 1/3, and a cold brew with probability 1/6. What is the probability that exactly 1 of each drink (cappuccino, espresso, and cold brew) have been sold at the end of T hours?

c) Unfortunately, Café 126 is located adjacent to their competitor, Sunducks. Customers enter Sunducks according to a Poisson Process at rate μ, independent of arrivals to Café 126. What is the probability that, at the time when Café 126 gets their kth customer, Sunducks will have had exactly n customers?

5 Confident Gambling [4 + 4 points]

It is suspected that a casino is cheating with a die. It is further believed that the probability for a roll resulting in a 6 is set to p, and the probability for each of the other outcomes (1, 2, 3, 4 or 5) is set to $(1-p)/5$. To investigate this, we first estimate p after N rolls by $(X_1 + X_2 + ... + X_N)/N$, where $X_i = 1$ if roll number i has the outcome of 6, and $X_i = 0$, otherwise. We roll the die 100 times, and observe a 6 on 25 of those rolls.

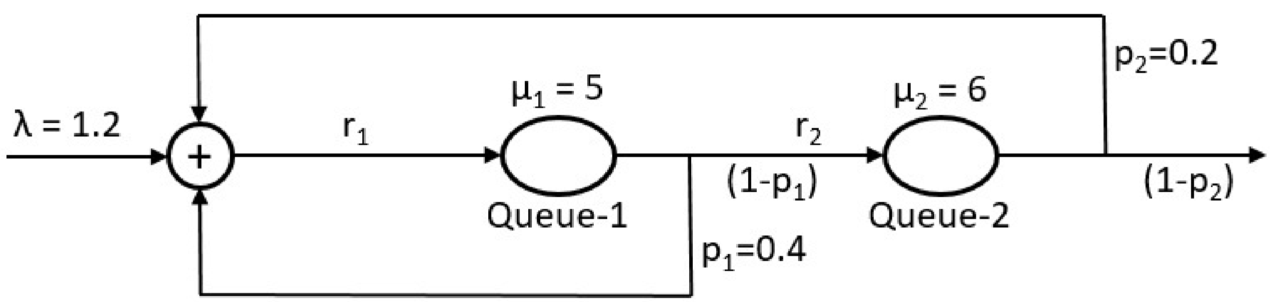
Hints/Facts: The variance of a Bernoulli random variable is at most 1/4. Assume $\Phi^{-1}(0.95) = 1.65$, $\Phi^{-1}(0.975) = 2$ and $\Phi^{-1}(0.99) = 2.32$, where Φ is the CDF of $\mathcal{N}(0,1)$ distribution.

a) Find the 95% confidence interval for p by indicating the numerical values of the endpoints of the interval.

b) Suppose you want to be sure that the true value of p is within 0.01 of the estimate with at least 0.95 probability. Approximate the minimum value of N required.

6 Jackson [2 + 3 + 3 + 3 + 2 points]

Consider the Jackson network shown in the figure below. Jobs arrive to the network according to a Poisson process with rate $\lambda = 1.2$ jobs/s, and the exponential service rates at Queue-1 and Queue-2 are $\mu_1 = 5$ and $\mu_2 = 6$ jobs/s, respectively. Also, $p_1 = 0.4$ and $p_2 = 0.2$ are the probabilities for a job to rejoin the first queue after service at Queue-1 and Queue-2, respectively. Note that from the time a job enters the network until it leaves the network, it's always at one of the two queues, i.e., routing occurs in zero time.



a) Find the rates r_1 and r_2, the total rates at which jobs enter the two queues, respectively.

b) Find the invariant probability that there are no jobs in the entire network.

c) Find the average number of total jobs in the network under the invariant distribution.

d) What's the average delay for a job (duration from the arrival to and departure from the network) assuming the network is operational for a long time?

e) Is the two-dimensional CTMC associated with this Jackson network reversible? Justify your answer.