EECS 126 Probability and Random Processes | UC Berkeley, Spring 2021
Thomas Courtade | April 6, 2021

Midterm 2

Last Name	First Name	SID

***Rules*.**

- **Unless otherwise stated, all your answers need to be justified and your work must be shown. Answers without sufficient justification will get no credit.**
- You have 70 minutes to complete the exam and 10 minutes exclusively for submitting your exam to Gradescope. (DSP students with $X\%$ time accomodation should spend $70 \cdot X\%$ time on the exam and 10 minutes to submit).
- Collaboration with others is strictly prohibited.
- You should not discuss the exam with anyone (this includes your roommate, your parents, social media, reddit, etc.) until 24 hours after the exam concludes (April 7th, 2:10pm).
- You may reference your notes, the textbook, and any material that can be found through the course website. You may use Google to search for general knowledge or use calculators. However, searching for a question is not allowed.
- For any clarifications you have, please create a private Piazza post. We will have a Google Doc that shows our official clarifications.

Problem	points earned	out of
Honor Code		5
Problem 1		15
Problem 2		21
Problem 3		15
Problem 4		26
Problem 5		18
Total		100

Honor Code [5 points]

Please copy the following word for word, and sign afterwards.

By my honor, I confirm that

1. *this work is my own original work;*
2. *I have not and will not discuss this exam with anyone during the exam and for 24 hours after the exam;*
3. *I have not and will not Google/search for any of these exam problems.*

1 Ya∗tze∗ [5 + 10 points]

Consider a 6-sided die, with faces numbered 1-6. The die is weighted so that you are twice as likely to roll an even number as an odd number. Among even numbers, all outcomes are equally likely. Similarly, among odd numbers, all outcomes are equally likely.

(a) Suppose you roll the die $k \gg 1$ times. Approximately how many bits per outcome, on average, are needed to describe the sequence of k independent rolls?

(b) Let $(n_k)_{k\geq 1}$ be a sequence of integers satisfying $\lim_{k\to\infty} k/n_k = B$, where the *bandwidth* $B \geq 0$ is a fixed constant. You would like to communicate the sequence of k rolls from part (a) over n_k uses of a Binary Erasure Channel with erasure probability $p \in [0,1]$ (i.e., a BEC(p)). If you want to make the probability of miscommunication arbitrarily small in the limit as $k \to \infty$, what is the required relationship between p and B?

[Let A denote your answer to part (a), and write your answer in terms of A.]

(a) By the source coding theorem, we need approximately $H(X)$ bits on average to encode each outcome, for k large. The probability of rolling an even number is $2/9$, and the probability of rolling an odd number is $1/9$. Hence, we need

$$-3 \times 2/9 \log(2/9) - 3 \times 1/9 \log(1/9) \approx 2.5\text{bits}$$

to encode each roll. It is also fine to say that we need approximately $kH(X) \approx 2.5 \times k$ bits to describe the sequence.

(b) By the channel coding theorem, we know that information can be reliably sent at any rate

$$\frac{\#\text{ message bits}}{\#\text{ channel uses}} = R < C = 1 - p,$$

and any rate exceeding C will incur errors with probability approaching 1. In this case, our number of message bits is $kA + o(k)$, and our number of channel uses is n_k. So, we find the relationship

$$\frac{kA + o(k)}{n_k} < (1 - p).$$

As $k \to \infty$, we have the necessary (and sufficient) relationship $AB < (1 - p)$.

2 Golden Bear Bets [5 + 7 + 9 points]

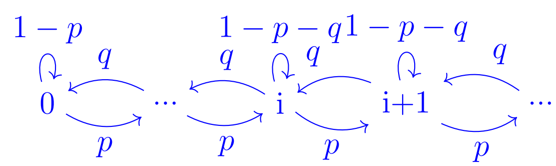
Inspired by your GameStop gains, you have bought a share of OSKI stock. This time, you decide to model stock prices with a Markov chain. You believe that every day, prices will increase by \$1 with probability p, decrease by \$1 with probability q (provided the price is \$1 or more), or stay the same. Assume prices take non-negative integer values and $p + q \leq 1$.

(a) Draw out the state transition diagram. Is this chain irreducible?

(b) Under what conditions on p and q is the chain positive recurrent? **Briefly** justify your answer.

(c) Assume $p = q > 0$ and $k \geq 10$. What is the probability that the stock price will reach $k + 30$ before it falls to $k - 10$, if we are starting at state k?

(a) The chain is irreducible provided $p, q > 0$, otherwise it is reducible.



(b) $0 < p < q$. It is a birth-death chain, so it is positive recurrent if the forward rate is less than the backwards rate.

[We didn't expect you to address the following corner cases, but technically speaking, picking $p = q = 0$ also gives a (reducible) chain in which all states are positive recurrent. However, this is a degenerate situation where the chain never leaves its starting state. Choosing $p = 0$ ensures state 0 is positive recurrent, but the other states are transient unless $q = 0$ also.]

(c) **Solution 1:** This is precisely the Gambler's ruin problem, so using the hitting probabilities computed in lecture, the probability is $\frac{10}{10+30} = \frac{1}{4}$.

Solution 2: We can also solve the FSE directly. Let $\alpha(n) = P(T_{n+30} < T_{n-10} | X_0 = n)$ be the probability of hitting state $n + 30$ before $n - 10$. We can write

$$\alpha(p - 10) = 0$$

$$\alpha(n + 30) = 1$$

$$\alpha(x) = \frac{\alpha(x+1) + \alpha(x-1)}{2}, \forall x = n - 9, n - 8, ..., n + 29$$

Notice that $\alpha(x)$ is exactly the midpoint between $\alpha(x - 1)$ and $\alpha(x + 1)$, so the graph of $\alpha(x)$ must be a straight line with endpoints $(n - 10, 0)$ and $(n + 30, 1)$. n is a quarter of the way between $n - 10$ and $n + 30$, so $\alpha(n)$ is a quarter of the way between 0 and 1. Thus, $\alpha(n) = \frac{1}{4}$.

Solution 3: Another way to see this is by noting the expected price of the stock on day $n+1$, X_{n+1}, is the same as day k, i.e. $\mathbb{E}[X_{n+1}] = \mathbb{E}[X_n] = k$, because $p = q$. Furthermore, consider $k-10$ and $k+30$ as sink states. As $n \to \infty$, X_n must be equal to either $k-10$ or $k+30$ almost surely: say it is equal to $k+30$ with probability x. Since $\mathbb{E}[X_n] = k$, we have $k = (1-x)(k-10) + x(k+30)$. Solving yields $x = \frac{1}{4}$.

3 Maximizing Uptime [5 + 10 points]

Aditya has two wheels for his unicycle (the second wheel is a backup wheel in case the one in use goes flat, so only one wheel is ever used at a time). When a wheel is being used, the time until it goes flat is exponentially distributed with rate μ. When a wheel goes flat, the other one, if it is not flat, immediately starts being used; additionally, the flat one is taken immediately to the repair shop, where the repair time is exponentially distributed with rate λ (ignore the time it might take to bring it to the repair shop). Assume that only one wheel can be serviced/repaired at a time.

(a) Model the number of operational wheels at time $t \geq 0$ as a CTMC. In particular, draw the transition diagram with states/arrows labeled appropriately, and explicitly state the Q-matrix.

(b) Find the long run probability that Aditya is unable to use his unicycle – i.e., both wheels are flat.

(a) Consider state space $\{0, 1, 2\}$, where the state represents the number of operational wheels. From the problem description, we have the following rate matrix and transition diagram:

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 \\ \mu & -(\lambda+\mu) & \lambda \\ 0 & \mu & -\mu \end{bmatrix}.$$

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 2 \arrow[l, bend left, "\mu"] \end{tikzcd}$$

(b) We need to solve $\pi Q = 0$ to find the long run probability $\pi(0)$ that no wheels are operational. Setting inner product of π with the first and third columns of Q equal to zero gives

$$\lambda\pi(0) = \mu\pi(1)$$
$$\lambda\pi(1) = \mu\pi(2).$$

Hence,

$$\pi(i) = \big(\lambda/\mu\big)^i \pi(0).$$

Since π is a probability vector, we must have

$$1 = \pi(0) + \pi(1) + \pi(2) = (1 + \lambda/\mu + (\lambda/\mu)^2)\pi(0) \quad \Rightarrow \quad \pi(0) = \frac{1}{1 + \lambda/\mu + (\lambda/\mu)^2}.$$

4 Poisson Jobs [6 + 6 + 6 + 8 points]

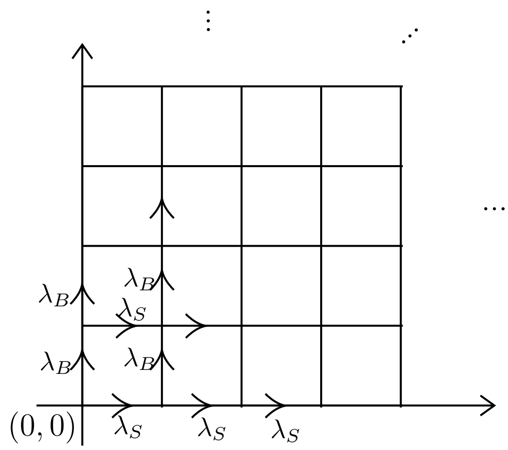


Figure 1: A CTMC where (X_t, Y_t) is the state at time t; vertices represent states.

Amazon receives deep learning jobs from Berkeley and Stanfurd according to independent Poisson processes with respective rates λ_B and λ_S. For $t \geq 0$, Let X_t denote the total number of jobs received from Stanfurd up to and including time t, and let Y_t denote the total number of jobs received from Berkeley up to and including time t. Although not needed for this problem, you can picture the pair (X_t, Y_t) as evolving according to a CTMC, as in Figure 1.

(a) At any given instant, what is the probability that the next three jobs received by Amazon are all from Berkeley?

(b) Suppose you inspect the system at some time $t \gg 0$, assumed to be infinitely far in the future. What is the expected time between the previous job and the next job?

(c) Suppose Amazon has received 2 jobs from Berkeley and 3 jobs from Stanfurd by time t. What is the expected time at which Amazon receives the 10th job from Berkeley?

(d) Again, suppose Amazon has received 2 jobs from Berkeley and 3 jobs from Stanfurd at time t. What is the expected number of jobs Amazon received from Stanfurd when it has received 10 jobs from Berkeley?

(a) **Solution 1:** By min of independent exponentials, the probability one job is from Berkeley is $\frac{\lambda_B}{\lambda_B+\lambda_S}$, so by the memoryless property of the Poisson process, the probability all three are from Berkeley is:

$$\left(\frac{\lambda_B}{\lambda_B+\lambda_S}\right)^3$$

Solution 2: We can view the combined job process as a merged Poisson process with rate $\lambda_S + \lambda_B$, and the jobs from Berkeley and Stanfurd can be obtained by splitting this process (by independently marking arrivals with probabilities $\frac{\lambda_B}{\lambda_B+\lambda_S}$ and $\frac{\lambda_S}{\lambda_B+\lambda_S}$, respectively). The probability the first three jobs are marked as Berkeley is thus

$$\left(\frac{\lambda_B}{\lambda_B + \lambda_S}\right)^3$$

(b) The expected time from time t until the next job is $\frac{1}{\lambda_S+\lambda_B}$ by min of exponentials. Since $t \gg 0$, by random incidence paradox, the expected time between the previous job and the next job is twice that, so the answer is $\frac{2}{\lambda_S+\lambda_B}$.

(c) The Poisson process of Berkeley jobs is independent of the process of Stanfurd jobs. The average time between jobs is $\frac{1}{\lambda_B}$, so the expected time until 10 jobs is $t + \frac{10-2}{\lambda_B} = t + \frac{8}{\lambda_B}$ by the memoryless property of the Poisson process.

(d) Let the time Amazon receives the 10th Berkeley job be T_B. Let S be the number of jobs from Stanfurd received between times t and T_B. By iterated expectation,

$$\mathbb{E}[S] = \mathbb{E}[\mathbb{E}[S|T_B]]$$

Since the Stanford job process is Poisson, independent of the Berkeley jobs, conditioned on T_B, we have $S \sim \text{Poisson}(\lambda_S(T_B - t))$, which has (conditional) expected value $\mathbb{E}[S|T_B] = \lambda_S(T_B - t)$. Thus, by part c,

$$\mathbb{E}[S] = \mathbb{E}[\lambda_S(T_B - t)] = \lambda_S(\mathbb{E}[T_B] - t) = \frac{8\lambda_S}{\lambda_B}.$$

Since Stanfurd already had 3 jobs by time t, the number they receive by time T_B is, in expectation,

$$3 + \mathbb{E}[S] = 3 + \frac{8\lambda_S}{\lambda_B}.$$

5 This Question is Almost Surely Solvable. Probably. [6 + 4 + 8 points]

(a) Let $(S_n)_{n\geq 1}$ be i.i.d. non-negative random variables, with finite mean $\mathbb{E}[S_n] = \mu < +\infty$. Consider an arrival process $(N_t)_{t\geq 0}$ (a continuous-time counting process, but not necessarily a Poisson process, which would require the S_n's to be exponentials) which has inter-arrival times equal to the S_n's. That is, the first arrival comes at time $T_1 = S_1$, the second arrival comes at time $T_2 = T_1 + S_2 = S_1 + S_2$, and so forth, so that the kth arrival comes at time $T_k = \sum_{n=1}^{k} S_n$.

Argue that $\lim_{t\to\infty} N_t = +\infty$ almost surely.

(b) For the setting of part (a), compute $\lim_{t\to\infty} \frac{N_t}{N_t+1}$ and describe its mode of convergence.

(c) For the setting of part (a), compute $\lim_{t\to\infty} N_t/t$, and describe its mode of convergence. [*Hint:* For each $t \geq 0$, we have $T_{N_t} \leq t \leq T_{N_t+1}$. Use this to "sandwich" N_t/t and apply an appropriate law of large numbers.]

(a) If there are a finite number of arrivals with positive probability, then this means that some interarrival time was necessarily infinite, and thus $\Pr(S_n = +\infty) > 0$. This contradicts the assumption that $\mathbb{E}[S_n] < +\infty$.

[The above explanation is sufficient, but if you prefer to see this worked out explicitly in terms of events and properties of probability measures, observe that

$$\{\lim_{t\to\infty} N_t < +\infty\} = \cup_{n\geq 1}\{S_n = +\infty\}.$$

Hence, by countable additivity and the union bound

$$\Pr\{\lim_{t\to\infty} N_t < +\infty\} = \Pr\left\{\cup_{n\geq 1}\{S_n = +\infty\}\right\} \leq \sum_{n\geq 1} \Pr\{S_n = +\infty\} = 0,$$

where the last equality follows since $\mathbb{E}[S_n] < +\infty$ (and thus $\Pr\{S_n = +\infty\} = 0$). Hence, $\lim_{t\to\infty} N_t = +\infty$ a.s.]

(b) Since $N_t \to +\infty$ a.s., we have that

$$\frac{N_t}{N_t+1} = 1 - 1/(N_t+1) \to 1 \quad a.s.$$

(c) Using the hint, note that

$$\frac{N_t}{N_t+1}\frac{N_t+1}{T_{N_t+1}} = \frac{N_t}{T_{N_t+1}} \leq \frac{N_t}{t} \leq \frac{N_t}{T_{N_t}}.$$

Since $N_t \to +\infty$ a.s., the SLLN implies that

$$\lim_{t\to\infty} \frac{T_{N_t+1}}{N_t+1} = \lim_{t\to\infty} \frac{T_{N_t}}{N_t} = \mu.$$

So, combined with part (b) and the sandwich inequality above, we have $\frac{N_t}{t} \to 1/\mu$ a.s.