EECS 126 Probability and Random Processes — UC Berkeley, Spring 2023
Thomas Courtade — March 23, 2023

Midterm 2

Last Name	First Name	SID

Left Neighbor First and Last Name	**Right Neighbor** First and Last Name

***Rules*.**

- **Write in your SID on every page to receive 1 point.**
- **Start this exam by doing Problem 1 first.**
- **Unless otherwise stated, all your answers need to be justified and your work must be shown. Answers without sufficient justification will get no credit.**
- You have 80 minutes to complete the exam. (DSP students with $X\%$ time accommodation should spend $80 \cdot X\%$ time on completing the exam).
- This exam is closed-book. You may reference one double-sided sheet of notes. No calculator or phones allowed.
- Remember the Berkeley Honor Code: "*As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others.*" Any violation of academic integrity will be taken seriously, and could result in disciplinary consequences.

Problem	out of
SID	1
Problem 1	6
Problem 2	13
Problem 3	18
Problem 4	20
Problem 5	20
Problem 6	22
Total	100

1 Answer This Question First [3 + 3 points]

a) What's the most interesting thing you've learned so far this semester (in this class, or any other)?

b) On a scale from 1-10 (10 being best), how well do you think you understand the material in this course?

2 Casino [8 + 5 points]

A casino has a new game where the host goes behind a curtain, flips a fair coin repeatedly, and records the number of tosses N it took to see the first heads. Your job, as the player of the game, is to determine the value of N by asking the host as few yes/no questions as possible.

a) How many yes/no questions, on average, do you need to ask in order to determine the value of N?

b) Give an optimal sequence of *simple* questions for determining N. By "optimal", we mean that you should ask fewest possible questions on average in order to determine N.

3 Simple Shuffles [6 + 8 + 4 points]

a) A square matrix P is called *doubly stochastic* if P is a stochastic matrix, and all columns sum to one. What is the stationary distribution of an irreducible finite-state DTMC with a doubly stochastic transition matrix P?

b) A robot shuffles a standard deck of 52 cards according to a simple algorithm: Starting with any initial ordering of the cards, a single card is uniformly chosen at random from the 52 cards in the deck, and is then re-inserted at another uniformly random location (it is possible for the card to be returned to the same place). This process is repeated over and over, ad infinitum. Argue that, as the number of these simple shuffles tends to infinity, the deck becomes perfectly shuffled.

c) If the deck in part (b) starts with ordering $(1, 2, \ldots, 52)$, how many operations will it take on average before the cards are again in the order $(1, 2, \ldots, 52)$?

4 Double Heads [10 + 10 points]

Suppose you flip a fair coin repeatedly.

a) Given that the first flip turns heads, what is the expected number of tosses (including your first toss) you need to make until you see two heads in a row?

b) Again, given that the first flip turns heads, what is the probability you see two heads in a row before you see two tails in a row?

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(Extra page for Problem 4)

5 Leaky Roof [6 + 6 + 8 points]

a) Let $(U_i)_{i\geq 1} \sim_{\text{i.i.d.}} \text{Unif}(0,1)$. Find $\mathbb{E}[\max\{U_1,\ldots,U_n\}]$.

b) Raindrops from a leaky roof fall into a small red bucket according to a Poisson process, having rate 1 drop/sec. You empty the red bucket into a larger blue bucket at a rate of once per minute, according to a Poisson process which is independent of the raindrop process. When you empty the red bucket for the first time, what is the probability that it contains 60 or more raindrops?

c) Assume the model of part (b). For a given time $t \geq 0$ (t has units of seconds), find the expected number of raindrops in the larger blue bucket. (Make the simplifying assumption that the buckets never overflow.) Hint: Use part (a).

(Extra page for Problem 5)

6 Waiting in Line [6 + 8 + 8 points]

Consider the M/M/1 queue with arrival rate $\lambda > 0$, and service rate $\mu > 0$. Recall that this refers to a queuing system for which the arrival process to the system is a rate-λ Poisson process, the service times are i.i.d. Exp(μ), and there is one server which serves customers on a first-in first-out basis.

a) Let $(X_t)_{t\geq 0}$ denote the number of people in the system at time $t \geq 0$. Draw the state transition diagram for this CTMC, with clearly labeled states and arrows labeled with nonzero transition rates.

b) Under what conditions does a stationary distribution exist? Assuming these conditions are met, what is the stationary distribution?

c) You are a customer who arrives to the system after it's been running for a very long time (i.e., assume the system is in steady-state). What is the expected time you spend in the system before departing (total time spent waiting in the queue, and in service)?

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(Extra page for Problem 6)

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