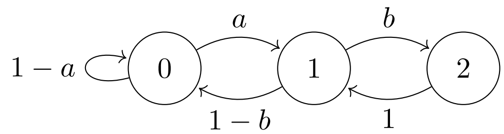
37. **Three-State Chain**

Consider the following Markov chain, where $0 < a, b < 1$.



a. Calculate $\mathbb{P}(X_1 = 1, X_2 = 0, X_3 = 0, X_4 = 1 \mid X_0 = 0)$.

b. Show that the Markov chain is irreducible and aperiodic.

c. Find the invariant or stationary distribution.

Solution:

a. By the Markov property, this probability is

$$P(0,1) \cdot P(1,0) \cdot P(0,0) \cdot P(0,1) = a \cdot (1-b) \cdot (1-a) \cdot a = a^2(1-a)(1-b).$$

b. The chain is irreducible because its transition diagram is strongly connected — there is a path from any state to any other state — and it is aperiodic because there is a self-loop.

c. To find the stationary distribution, let us solve the balance equations:

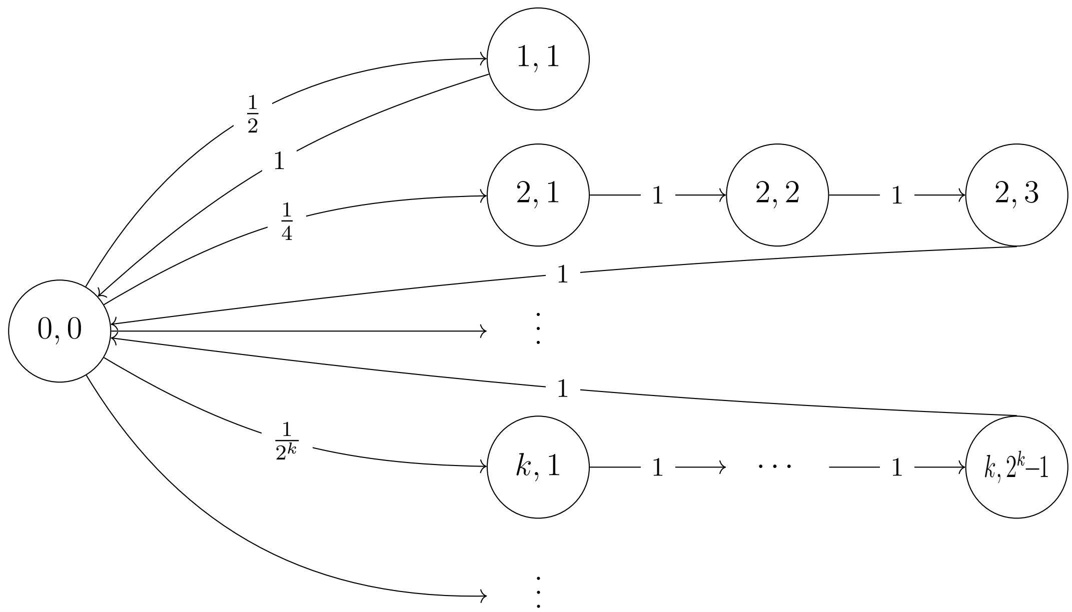
$$\pi(2) = b\pi(1),\ \pi(1) = a\pi(0) + \pi(2),\ \pi(0) + \pi(1) + \pi(2) = 1,$$

from which we find the solution

$$\begin{bmatrix}\pi(0) & \pi(1) & \pi(2)\end{bmatrix} = \frac{1}{1-b+a+ab}\begin{bmatrix}1-b & a & ab\end{bmatrix}.$$

6 Bot on a Stroll [30 points]

The EECS 126 Bot is taking a walk on a Markov chain with state space $\mathbb{N} \times \mathbb{N}$, starting from state $(0, 0)$, as shown by the graph below.



From state $(0, 0)$, the bot chooses "path k" with probability 2^{-k} for $k = 1, 2, \ldots$ Each path k contains $2^k - 1$ states, which the bot will travel through in sequence then return to $(0, 0)$ deterministically.

(a) Is this Markov chain irreducible? Justify your answer.

(b) What is the period of this Markov chain?

(c) What is the expected time to return to state $(0, 0)$?

(d) Is this Markov chain positive recurrent, null recurrent, or transient? Justify your answer.

(a) The Markov chain is irreducible. The path from state (x_t, y_t) to (x_{t+1}, y_{t+1}) can be traced out by first following path x_t to reach $(0, 0)$ in $2^{x_t} - y_t$ steps, then jumping to $(x_{t+1}, 1)$ and following path x_{t+1} for $y_{t+1} - x_{t+1}$ steps.

(b) The period is 2 since the the time to travel path k is 2^k for $k = 1, 2, 3, \ldots$ with a GCD of 2.

(c) The probability to take path i is 2^{-i}, and the time it takes to return to $(0, 0)$ given that we take path i is $(2^i - 1) + 1 = 2^i$. Thus, the expected return time is

$$\sum_{i=1}^{\infty} 2^i \cdot 2^{-i} = \sum_{i=1}^{\infty} 1 = \infty.$$

 Student ID (write on every page for 1 point): ______________

(d) The Markov chain is null recurrent. From $(0,0)$, no matter which path i the bot takes, it will always return to $(0,0)$ in $2^i < \infty$ steps. Since it always returns to state $(0,0)$, the Markov chain is recurrent. From the previous part, since the expected return time is infinite, we further classify the Markov chain as null recurrent.

1 Scheduling Conflict [30 points]

Oh no, EECS 126 and CS 170 are having Homework Parties at Cory Courtyard at the same time! EECS 126 and CS 170 students arrive at Cory Courtyard independently according to two Poisson processes with rates λ_{126} and λ_{170} respectively. For simplicity, assume no student is taking the two classes at the same time.

(a) Let T_3 be the time that the third student arrives. What is $\mathrm{E}[T_3]$?

Andy walks by Cory Courtyard at time t and sees there are three students.

(b) What is the expected time between the last student arrival before Andy and the next student arrival after Andy?

(c) What is the probability that Andy sees more EECS 126 students than CS 170 students when he walks by?

(a) We can merge the two Poisson processes into one student arrival process of rate $\lambda_{126} + \lambda_{170}$. Since the interarrival times are independent, and each interarrival interval has an expected length of $(\lambda_{126} + \lambda_{170})^{-1}$, we have

$$\mathrm{E}[T_3] = \frac{3}{\lambda_{126} + \lambda_{170}}.$$

(b) This is slightly different from the Random Incidence Property since we are given there are three arrivals before time t, but a similar analysis technique can apply. The three arrivals before time t are distributed as the order statistics of three independent uniform random variables in $[0, t]$. The three student arrivals split the interval $[0, t]$ into four sub-intervals, each of which has the same expected length, so

$$\mathrm{E}[t - T_3 \mid N_t = 3] = \frac{1}{4}t.$$

Then the next student arrival takes $(\lambda_{126} + \lambda_{170})^{-1}$ time in expectation, giving us the answer

$$\frac{1}{4}t + \frac{1}{\lambda_{126} + \lambda_{170}}.$$

(c) By competing Exponentials, each student that arrives has probability $p := \lambda_{126}/(\lambda_{126} + \lambda_{170})$ of being a EECS 126 student, independent of all other students. Thus, the number of EECS 126 students in the first three arrivals is distributed as $\mathrm{Binomial}(3, p)$. Then the probability that two or more EECS 126 students are in the first three student arrivals is

$$\frac{3\lambda_{126}^2\lambda_{170} + \lambda_{126}^3}{(\lambda_{126} + \lambda_{170})^3}.$$