UC Berkeley
Department of Electrical Engineering and Computer Science

EE 126: PROBABLITY AND RANDOM PROCESSES

Solutions 6
Spring 2006

Problem 6.1

Solution: It is useful to think of X as a uniform random variable in $[\frac{\pi}{2}; \frac{3\pi}{2}]$, we have

$$F_X(x) = \begin{cases} 0, & z \leq \frac{\pi}{2} \\ \frac{x}{2\pi}, & \frac{\pi}{2} \leq x \leq \frac{3\pi}{2} \\ 1, & x \geq \frac{3\pi}{2} \end{cases}$$

For a particular value $y \in [-1; 1]$ of the random variable Y , we have two values x_1 x_2 of the random variable X corresponding to it. Requiring $P(\sin(X) \leq y)$ is equivalent to requiring $P(X \leq x_1) + P(X \geq x_2)$. By symmetry of the $\sin(y)$ function, $P(X \leq x_1) = P(X \geq x_2)$. See Figure 1.

Thus,

$$\begin{aligned} P(Y \leq y) &= P(\sin(X) \leq y) \\ &= 2P(X \leq \sin^{-1}(y)) \\ &= 2\sin^{-1}(y) \end{aligned}$$

for $y \in [-1; 1]$. From which,

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} P(Y \leq y) \\ &= \frac{d}{dy} 2\sin^{-1}(y) \\ &= \frac{1}{\pi\sqrt{(1-y^2)}} \end{aligned}$$

for $y \in [-1; 1]$

Problem 6.2

Solution:

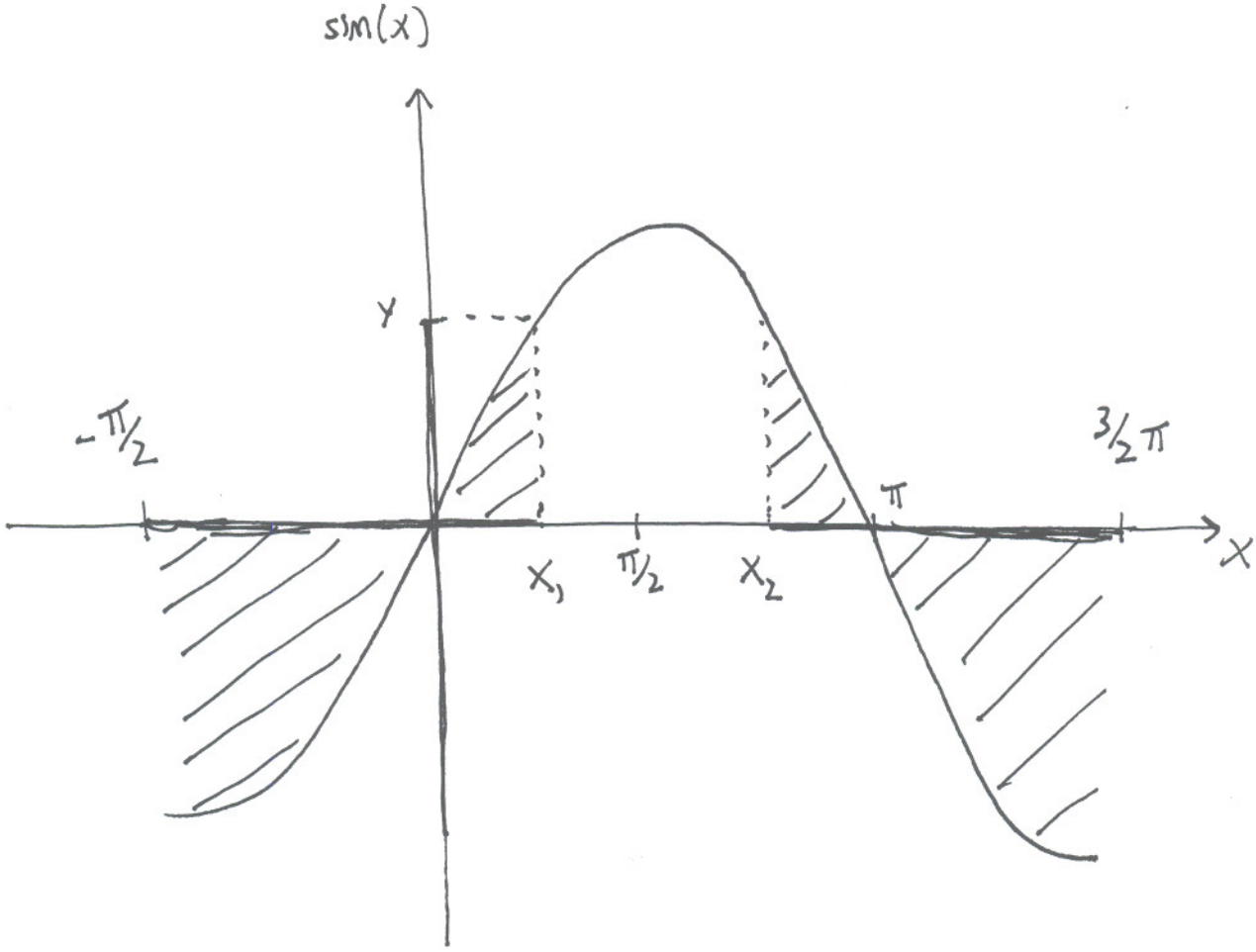
Let G denote the event that Dino has a good day and let B denote the event that Dino has a bad day. We are given that $P(G) = P(B) = 0.5$. Let T be the time it takes Dino to cook a souffle, so that $f_{T|G}$ is uniform between 1/2 and 1, and $f_{T|B}$ is uniform between 1/2 and 3/2. We need to find $P(B \mid T \leq 3/4)$.

Using Bayes's rule, we have

$$P(B \mid T \leq 3/4) = \frac{P(T \leq 3/4 \mid B)P(B)}{P(T \leq 3/4)} = \frac{P(T \leq 3/4 \mid B)P(B)}{P(T \leq 3/4 \mid B)P(B) + P(T \leq 3/4 \mid G)P(G)}$$

Evaluating this expression, we find that

$$P(B \mid T \leq 3/4) = \frac{(\frac{1}{4})(\frac{1}{2})}{(\frac{1}{4})(\frac{1}{2}) + (\frac{1}{2})(\frac{1}{2})} = \frac{1}{3}.$$



$$\sin(x) \le y \quad \Rightarrow \quad x \le x_1 \text{ and } x \ge x_2$$

Figure 1

Problem 6.3

(a) Let S and R be the events that the day is sunny and rainy, respectively. We are given that $f_{X|S}$ and $f_{X|R}$ are uniform PDFs over the intervals $[30, 45]$ and $[40, 60]$, respectively. Moreover we know that $\mathbf{P}(S) = 2/3$ and $\mathbf{P}(R) = 1/3$. If follows that

$$\begin{aligned} f_X(x) &= \mathbf{P}(S) f_{X|S}(x) + \mathbf{P}(R) f_{X|R}(x) \\ &= \begin{cases} (2/3)(1/15) = 0.0444, & \text{if } 30 \le x < 40, \\ (2/3)(1/15) + (1/3)(1/20) = 0.0611, & \text{if } 40 \le x \le 45, \\ (1/3)(1/20) = 0.0167, & \text{if } 45 < x \le 60\,. \end{cases} \end{aligned}$$

As for the expected value and the variance of X, we have

$$\mathbf{E}[X] = \mathbf{E}[X \mid S]\mathbf{P}(S) + \mathbf{E}[X \mid R]\mathbf{P}(R) = \frac{30+45}{2} \cdot \frac{2}{3} + \frac{40+60}{2} \cdot \frac{1}{3} = \frac{125}{3} = 41.67,$$

$$\begin{aligned} \mathbf{E}[X^2] &= \mathbf{E}[X^2 \mid S]\mathbf{P}(S) + \mathbf{E}[X^2 \mid R]\mathbf{P}(R) \\ &= \frac{2}{3} \cdot \frac{1}{15} \int_{30}^{45} x^2\, dx + \frac{1}{3} \cdot \frac{1}{20} \int_{40}^{60} x^2\, dx \\ &= 1794.4, \end{aligned}$$

and

$$\text{var}(X) = 1794.4 - (41.67)^2 = 58.01.$$

(b)

$$\mathbf{P}(R \mid X = 45) = \frac{\mathbf{P}(R) f_X(45 \mid R)}{f_X(45)} = \frac{(1/3)(1/20)}{0.0611} = 0.2733.$$

(c) Let V be the average speed. We have $V = 20/X$, so that

$$\begin{aligned} f_V(v) &= f_X\left(\frac{20}{v}\right) \left| \frac{d}{dv}\left(\frac{20}{v}\right) \right| \\ &= \frac{20}{v^2} f_X\left(\frac{20}{v}\right) \\ &= \begin{cases} 1/(3v^2), & \text{if } 20/60 \le v < 20/45, \\ 55/(45v^2), & \text{if } 20/45 \le v \le 20/40, \\ 8/(9v^2), & \text{if } 20/40 < v \le 20/30, \end{cases} \end{aligned}$$

$$\mathbf{E}[V] = \int_{20/60}^{20/45} \frac{v}{2v^2} dv + \int_{20/45}^{20/40} \frac{25v}{18v^2} dv + \int_{20/40}^{20/30} \frac{8v}{9v^2} dv = 0.496,$$

$$\mathbf{E}[V^2] = \int_{20/60}^{20/45} \frac{v^2}{2v^2} dv + \int_{20/45}^{20/40} \frac{25v^2}{18v^2} dv + \int_{20/40}^{20/30} \frac{8v^2}{9v^2} dv = 0.253,$$

and

$$\text{var}(V) = 0.253 - (0.496)^2 = 6.984 \cdot 10^{-3}.$$

Problem 6.4

(a) We have

$$\mathbf{P}(B \mid A) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)} = \frac{\mathbf{P}(X < Y \leq 0.5)}{1 - \mathbf{P}(Y \geq 0.5)} = \frac{0.25}{1 - 0.25} = \frac{1}{3}.$$

(b) We have

$$f_{X|Y}(x \mid 0.5) = \frac{f_{X,Y}(x, 0.5)}{f_Y(0.5)} = \begin{cases} 2, & \text{if } 0 < x \leq 0.5, \\ 0, & \text{otherwise.} \end{cases}$$

The conditional expectation is $1/4$ and the conditional variance is $(0.5)^2/12 = 1/3$.

(c) We have

$$\begin{aligned} f_{X,Y|B}(x, y) &= \begin{cases} f_{X,Y}(x, y)/\mathbf{P}(B), & \text{if } (x, y) \in B, \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 4, & \text{if } (x, y) \in B, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore,

$$\begin{aligned} f_{X|B}(x) &= \int_0^1 f_{X,Y|B}(x, y)\, dy \\ &= \begin{cases} \displaystyle\int_x^{1-x} 4\, dx = 4(1 - 2x), & \text{if } 0 < x \leq 0.5, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

(d) We have

$$\mathbf{E}[XY] = \int_0^1 \int_0^{1-x} 2xy\, dy\, dx = \int_0^1 x(1 - x)^2\, dx = \frac{1}{12}.$$

(e) Let $Z = Y/X$. If $z \geq 0$,

$$f_Z(z) = \frac{d}{dz}\mathbf{P}\left(\frac{Y}{X} \leq z\right) = \frac{d}{dz}\left(\frac{z}{z+1}\right) = \frac{1}{(z+1)^2}.$$