UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: PROBABILITY AND RANDOM PROCESSES

Problem Set 3
Spring 2016

Issued: Thursday, February 4, 2016 **Due:** 9am, Thursday, February 11, 2016

Problem 1. In class, we discussed the Buffon's needle experiment. Let's explore a small extension here. A needle of length ℓ is dropped randomly on a plane surface that is partitioned in rectangles by horizontal lines that are a apart and by vertical lines that are b apart. Suppose that $\ell < a$ and $\ell < b$. What is the expected number of rectangle sides crossed by the needle? What is the probability that the needle crosses at least one side of a rectangle?

Problem 2. A scientist takes measurements of cell activity at discrete time intervals. Let X_i denote the measurement taken at time i, where the X_i are IID continuous random variables with pdf $f_X(x)$. We call X_k a record if $X_k > X_i$ for all $i < k$.

(a) Find the probability that X_n is a record where $n \geq 1$.

(b) What is the expected number of records in the first m trials? What happens as $m \to \infty$?

(c) Now, let $T = \min\{n > 1 : X_n \text{ is a record}\}$. In other words, T is the first record in the sequence of measurements. Find $P(T > n)$.

(d) Show that if Y is a discrete random variable that takes on non-neggative integer values, then $E[Y] = \sum_{i=0}^{\infty} P(Y > i)$ and use this to find $E[T]$.

(e) Give an intuitive interpretation of what parts b. and d. are telling us.

Problem 3. To efficiently sell advertising space, Google uses a (generalized) second-price auction[1]. In a second-price auction, the bidder with the highest bid wins the item, but only pays as much as the second highest bidder offered. Suppose we have a seller, Google, and n buyers. The buyer's bids are IID exponentials, where buyer i bids $X_i \sim \text{Exp}(\lambda)$. Let P be the price of the item. Find the distribution of P.

[1] It's true! See example 1 on https://support.google.com/adsense/answer/160525?hl=en

Problem 4. n graduate students at a Berkeley Research Lab come to work by bicycles every day, parking their bikes in an unlocked bike rack in their lab. On a given day, after a hard day's work, the students begin leaving one by one, taking their bikes out of the rack on their way out. The first student has forgotten what his bike looks like, and picks up one of the n bikes from the rack uniformly at random. Subsequently, the other $(n-1)$ students who leave one by one, follow the "honest if possible" policy of picking up their own bike if it is there, else picking a bike uniformly at random from the remaining collection in the rack.

(a) What is the probability that the first student finds his/her own bike?

(b) List the bikes that the last student may possibly take. What is the probability that the last student finds his/her own bike?

(c) List the bikes that the ith student may possibly take (for $i = 2, \ldots, n-1$). What is the probability that the this student finds his/her own bike?

(d) What is the expected number of students who go home with their own bikes? (What is this as n gets large)?

Problem 5. Consider random variables X and Y which have a joint PDF uniform on the triangle with vertices at $(0,0), (1,0), (0,1)$.

(a) Find the joint PDF of X and Y.

(b) Find the marginal PDF of Y.

(c) Find the conditional PDF of X given Y.

(d) Find $E[X]$ in terms of $E[Y]$

(e) Find $E[X]$.

Problem 6. Figure 1 shows the joint density $f_{X,Y}$ of random variables X and Y.

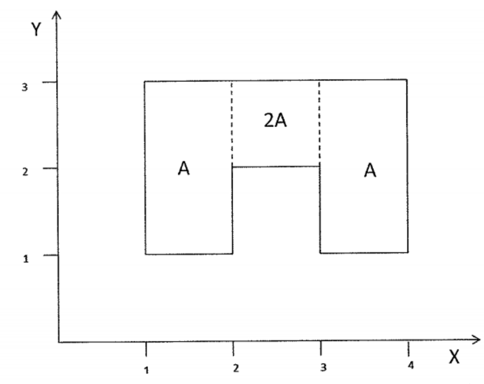

Figure 1: Joint pdf of X and Y.

(a) Find A and sketch f_X, f_Y and $f_{X|X+Y\leq 3}$.

(b) Find $\mathbb{E}[X|Y=y]$ for $1 \leq y \leq 3$ and $\mathbb{E}[Y|X=x]$ for $1 \leq x \leq 4$.

(c) Find $\text{cov}(X,Y)$.