UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: PROBABILITY AND RANDOM PROCESSES

Problem Set 4
Spring 2016

Issued: Thursday, February 18, 2016 **Due:** 9:00am Thursday, February 25, 2016

Problem 1. Midterm 01.

Problem 2. Consider a random variable Z with transform:

$$M_Z(s) = \frac{a - 3s}{s^2 - 6s + 8}$$

(a) Find the numerical value for the parameter a.

(b) Find $P(Z \geq 0.5)$.

(c) Find $E[Z]$ by using the probability distribution of Z.

(d) Find $E[Z]$ by using the transform of Z and without explicitly using the probability distribution of Z.

(e) Find $\text{Var}(Z)$ by using the probability distribution of Z.

(f) Find $\text{Var}(Z)$ by using the transform of Z and without explicity using the probability distribution of Z.

Problem 3. Let X, Y, and Z be independent random variables. X is Bernoulli with $p = 1/4$. Y is exponential with parameter 3. Z is Poisson with parameter 5.

(a) Find the transform of $5Z + 1$.

(b) Find the transform of $X + Y$.

(c) Consider the new random variable $U = XY + (1 - X)Z$. Find the transform associated with U.

Problem 4. In class, we learned some inequalities such as the Markov inequality, the Chebyshev inequality, and the Chernoff bound. In this problem, we will derive an inequality, which is a special case of Chernoff bound, using a simple counting method.
Suppose $X_1, \ldots, X_n$ are i.i.d. Bernoulli random variables with $\Pr(X_i = 1) = 1/2$.

(a) First, use the Chebyshev inequality to show that for any $\epsilon > 0$,

$$\Pr\left(\sum_{i=1}^{n} X_i \geq \frac{n}{2}(1+\epsilon)\right) \leq \frac{1}{\epsilon^2 n}. \tag{1}$$

The special case of Chernoff bound that we will derive is as follows: for any $\epsilon > 0$,

$$\Pr\left(\sum_{i=1}^{n} X_i \geq \frac{n}{2}(1+\epsilon)\right) \leq \exp\{-\frac{\epsilon^2 n}{10}\}. \tag{2}$$

We will derive (2) in the next steps. We should notice that if $\epsilon > 1$, we have $\Pr(\sum_{i=1}^{n} X_i \geq \frac{n}{2}(1+\epsilon)) = 0$. Therefore, we only need to consider the cases when $0 < \epsilon \leq 1$.

(b) Let M be the event that $X_1 = X_2 = \cdots = X_m = 1$, $m < n$. Show that for an integer k ($m \leq k \leq n$),

$$\Pr(M|\sum_{i=1}^{n} X_i = k) \geq (\frac{k-m}{n-m})^m,$$

and further, show that

$$\Pr(M|\sum_{i=1}^{n} X_i \geq k) \geq (\frac{k-m}{n-m})^m.$$

(c) For simplicity, we assume that $\frac{\epsilon n}{4}$ is an integer and let $m = \frac{\epsilon n}{4}$. Let G be the event that $\sum_{i=1}^{n} X_i \geq \frac{n}{2}(1+\epsilon)$. Show that

$$\Pr(M|G) \geq (\frac{1}{2} + \frac{\epsilon}{4})^m.$$

(d) Show that $\Pr(M) \geq \Pr(G)\Pr(M|G)$. Then show that

$$\Pr(G) \leq (1 + \frac{\epsilon}{2})^{-m}.$$

(e) Combining the fact that for any $0 < \epsilon \leq 1$,

$$\ln(1 + \frac{\epsilon}{2}) > \frac{2}{5}\epsilon, \tag{3}$$

show that (2) holds. (You do not need to prove (3).)

(f) Compare (1) and (2) and argue why the Chernoff bound is better than the Chebyshev inequality.