UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Problem Set 6
Spring 2016

Issued: Thursday, March 3, 2016 **Due:** 11:59pm, Thursday, March 10, 2016

Problem 1. Consider the Markov chain with state X_n, $n \geq 0$, shown in Figure 1, where $\alpha, \beta \in (0, 1)$.

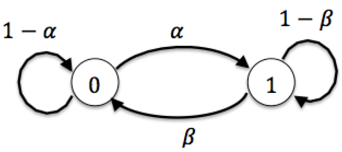

Figure 1: Markov chain for Problem 1

(a) Find the probability transition matrix P and the invariant distribution π of the Markov chain.

(b) Find two real numbers λ_1 and λ_2 such that there exists two non-zero vectors u_1 and u_2 such that $Pu_i = \lambda_i u_i$ for $i = 1, 2$. Further, show that P can be written as $P = U\Lambda U^{-1}$, where U and Λ are 2×2 matrices and Λ is a diagonal matrix.

Hint: This is called the eigendecomposition of a matrix.

(c) Find P^n in terms of U and Λ.

(d) Assume that $X_0 = 0$. Use the result in part (c) to compute the PMF of X_n for all $n \geq 0$. Verify that it converges to the invariant distribution.

Problem 2. A discrete-time Markov chain with seven states has the following transition probabilities:

$$p_{ij} = \begin{cases} 0.5, & (i,j) = (3,2), (3,4), (5,6) \text{ and } (5,7) \\ 1, & (i,j) = (1,3), (2,1), (4,5), (6,7) \text{ and } (7,5) \\ 0, & \text{otherwise} \end{cases}.$$

In the questions below, let X_k be the state of the Markov chain at time k.

(a) Give a pictorial representation of the discrete-time Markov chain.

(b) For what values of n is $\Pr(X_n = 5 \mid X_0 = 1) > 0$?

(c) What is the set of states $A(i)$ that are accessible from state i, for each $i = 1, 2, \ldots, 7$? Is the Markov chain irreducible?

(d) Identify which states are transient and which states are recurrent. For each recurrent state, state whether it is periodic (and give the period) or aperiodic.

(e) If $X_0 = 1$, what is the expected time for the Markov chain to reach state 7 for the first time?

Problem 3. You are playing a card game with your friend. You each have m cards in your hand. Out of the $2m$ total cards, m are green, and m are blue. At each round, you and your friend each randomly select a card from your respective hands and switch cards.

(a) Let X_n be the number of blue cards you have in your hand after you and your friend have exchanged cards n times. Find the transition probabilities for the Markov Chain X_n.

(b) A reversible Markov Chain has transition probabilities $Q_{ij} = \pi_j \frac{P_{ji}}{\pi_i}$ and a time reversible Markov Chain is such that $Q_{ij} = P_{ij}$. Show that the Markov Chain X_n is time reversible.

Problem 4. An ant is walking on the nonnegative integers. At each step, the ant moves forward one step with probability p, or slides back down to 0 with probability $1 - p$. What is the average time it takes for the ant to get to n?

Problem 5. (a) Find the steady-state probabilities $\pi_0, \ldots, \pi_{k-1}$ for the Markov chain in Figure 2. Express your answer in terms of the ratio $\rho = p/q$, where $q = 1 - p$. Pay particular attention to the special case $\rho = 1$.

(b) Find the limit of π_0 as k approaches infinity; give separate answers for $\rho < 1$, $\rho = 1$, and $\rho > 1$. Find limiting values of π_{k-1} for the same cases.

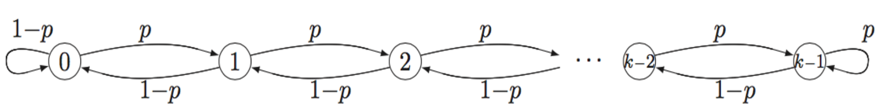

Figure 2: Markov chain for Problem 5