UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Problem Set 11

Spring 2016

Issued: Thursday, April 14, 2016 **Due:** 11:59PM Thursday, April 21, 2016

Problem 1. If $X = 0$, $Y = U[-1, 1]$ and if $X = 1$, $Y = U[0, 2]$. Solve a hypothesis testing problem so that the probability of false alarm is less than or equal β.

Problem 2. The random variables X, Y, Z are i.i.d. $\mathcal{N}(0, 1)$.

(a) Find $L[X^2 + Y^2|X + Y]$;

(b) Find $L[X + 2Y|X + 3Y + 4Z]$;

(c) Find $L[(X + Y)^2|X - Y]$.

Problem 3. Consider a photodetector in an optical communications system that counts the number of photons arriving during a certain interval. A user conveys information by switching a photon transmitter on or off. Assume that the probability of the transmitter being on is p. If the transmitter is on, the number of photons transmitted over the interval of interest is a Poisson random variable Θ with mean λ, and if it is off, the number of photons transmitted is 0. Unfortunately, regardless of whether or not the transmitter is on or off, photons may be detected due to "shot noise". The number N of detected shot noise photons is a Poisson random variable N with mean μ. Given the number of detected photons, find the LLSE of the number of transmitted photons.

Problem 4. Let $(V_n,\ n \geq 0)$ be i.i.d. $N(0, \sigma^2)$ and independent of $X_0 = N(0, u^2)$. Define

$$X_{n+1} = aX_n + V_n,\ n \geq 0.$$

(a) What is the distribution of X_n for $n \geq 1$?

(b) Find $E[X_{n+m}|X_n]$ for $0 \leq n < n + m$.

(c) Find u so that the distribution of X_n is the same for all $n \geq 0$.

Problem 5. The difficulty of an EE126 exam, Θ, is uniformly distributed on $[0, 100]$, and Alice gets a score X that is uniformly distributed on $[0, \Theta]$. Alice gets her score back and wants to estimate the difficulty of the exam.

(a) What is the LLSE for Θ?

(b) What is the MAP of Θ?

(c) Find the mean squared error of each estimate as a function of the score X.

Problem 6. The situation is the same as in the previous problem.

(a) What is the MMSE of Θ given X?

(b) Plot the MAP, LLSE, and MMSE as a function of the score X.

(c) Find the mean squared error of the MMSE as a function of the score X. Plot it along with the mean squared error of the MAP and LLSE.

Problem 7. Let the joint density of two random variables X and Y be

$$f_{X,Y}(x,y) = \frac{1}{4}(2x+y)1\{0 \le x \le 1\}1\{0 \le y \le 2\}.$$

First show that this is a valid joint distribution. Suppose you observe Y drawn from this joint density. Find MMSE$[X|Y]$.

Problem 8. Let X, Y, Z be three random variables. Prove formally that

$$E[|X - E[X|Y]|^2] \ge E[|X - E[X|Y,Z]|^2].$$

What is the intuition behind the inequality?