UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: PROBABILITY AND RANDOM PROCESSES

Problem Set 2
Spring 2017

Issued: Thursday, January 26, 2017 **Due:** 8am, Thursday, February 2, 2017

Problem 1. Let X be a random variable that takes values from 0 to 9 with equal probability 1/10.

(a) Find the pmf of the random variable $Y = X \bmod(3)$.

(b) Find the pmf of the random variable $Z = 5 \bmod(X + 1)$.

Problem 2. N couples enter a casino. After two hours, N of the original $2N$ people remain (the rest have left). Each person decides to leave with probability p independent of others' decisions. What is the expected number of couples still in the casino at the end of two hours?

Problem 3. Consider two strange countries, A and B. There are n cities with airports in country A and m cities with airports in country B. Let us call these cities $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_m$. The airports are such that no domestic flights are possible, i.e. there are no flights between (A_i, A_j) and (B_i, B_j). For each pair of cities in different groups, i.e., (A_i, B_j), there is a flight between these two cities with probability p, independently from all other pairs. An example of the flight connection $(n = 4, m = 4)$ is shown in Figure 1(a). Now, suppose a person lives in city A_1, and let $N_2(A_1)$ be the set of cities that this person can reach by taking *at most* 2 flights. We call $N_2(A_1)$ the two-flight neighborhood of A_1. An example of two-flight neighborhood is shown in Figure 1(b). What is the probability that there is at least one city in country A other than A_1 in $N_2(A_1)$, and at the same time, for every city in $N_2(A_1)$ other than A_1 itself, there is a *unique* flight route with at most 2 flights from A_1 to that city?

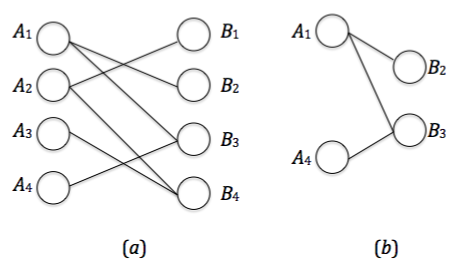

Figure 1: Flight connection and two-flight neighborhood of A_1.

Problem 4.
Suppose there is a 0-1 Bernoulli sequence $X^n = (X_1, X_2, \dots, X_n)$, where X_i's are i.i.d. Bernoulli random variables with $\Pr(X_i = 1) = p$. We define the *runs* of X^n as follows:
A subsequence $(X_i, X_{i+1}, \dots, X_j)$ of X^n is called a *run* if $X_i = X_{i+1} = \cdots = X_j$, $X_{i-1} \neq X_i$, and $X_{j+1} \neq X_j$. (Note that if $i = 1$, we do not need $X_{i-1} \neq X_i$, and if $j = n$, we do not need $X_{j+1} \neq X_j$.)
For example, there are 6 runs of $X^n = (0011001110001)$, i.e., $(00), (11), (00)$, (111), (000), and (1). What is the expected number of runs of the 0-1 Bernoulli sequence that we mentioned above?

Problem 5. Consider a random bipartite graph, G_1, with K left nodes and M right nodes. Each of the KM possible edges of this graph is connected with probability p independently. In the following problems, we consider the situations when M and K are large and Mp and Kp are constants.
Hint: Use the Poisson distribution to approximate binomial distribution.

(a) A singleton is a right node of degree one. As M and K get large, what is the expected number of left nodes that are connected to right nodes which are singletons?

(b) A doubleton is a right node of degree two. As M and K get large, what is the expected number of doubletons?

(c) We call 2 doubletons distinct if they are not connected to the same 2 left nodes. As K and M get large, what is the probability that two doubletons are distinct?

Problem 6. Consider the same setting as the previous problem.

(a) Let M_s be the number of doubletons for which both of the left nodes are also connected to singletons. Find $E[M_s]$ as K and M get large.

(b) We construct another random graph, G_2, as follows. Let K_s be the number of left nodes that are connected to singletons, which you calculated in part (a) of Problem 5. Graph G_2 has K_s nodes corresponding to these left nodes. Two nodes in G_2 are connected if there is a doubleton in G_1 that is connected to those left nodes. Thus, G_2 has M_s edges which you calculated in part (a). Argue that G_2 is equivalent to an Erdos-Renyi random graph.

(c) Although The Donald has very small hands, he is interested in 'yuge' components on the graph. A 'yuge' component is the largest set of nodes in the graph that is connected. An Erdos-Renyi random graph $G(N, q)$ has a 'yuge' component of size linear in N if $Nq > 1$. Suppose that $M = 4K$. Find a condition on p as a function of K such that G_2 has a 'yuge' component.