UC Berkeley
Department of Electrical Engineering and Computer Sciences

Electrical Engineering 126: Probability and Random Processes

Problem Set 3
Spring 2017

Issued: February 2, 2017 **Due:** 8 am, Thursday, February 9, 2017

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1. **Triangle Density**

   Let (X, Y) be uniformly distributed over the triangle with vertices $(0, 0)$, $(1, 0)$, and $(2, 1)$. Find the following:

   (a) $f_{X,Y}(x, y)$.

   (b) $f_X(x)$.

   (c) $E[Y \mid X = x]$.

2. **Graphical Density**

   Figure 1 shows the joint density $f_{X,Y}$ of the random variables X and Y.

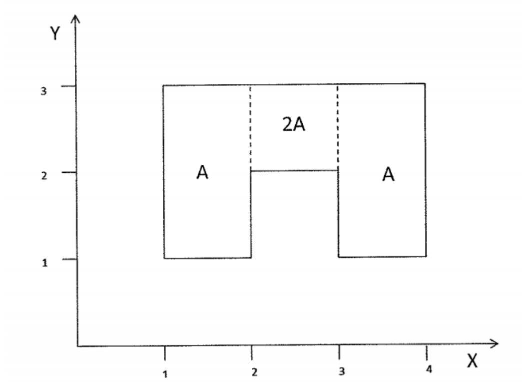

Figure 1: Joint density of X and Y.

   (a) Find A and sketch f_X, f_Y, and $f_{X|X+Y\leq 3}$.

   (b) Find $E[X \mid Y = y]$ for $1 \leq y \leq 3$ and $E[Y \mid X = x]$ for $1 \leq x \leq 4$.

   (c) Find $\text{cov}(X, Y)$.

3. **Office Hours**

   In an EE126 office hour, students bring either a difficult-to-answer question with probability $p = 0.2$ or an easy-to-answer question with probability $1-p = 0.8$. A GSI takes a random amount of time to answer a question, with this time duration being exponentially distributed with rate $\mu_D = 1$ (question per minute)-where D denotes difficult- if the problem is difficult, and $\mu_E = 2$ (questions per minute)-where E denotes easy-if the problem is easy.

(a.) You visit office hours and find a GSI answering the question of another student. Conditioned on the fact that the GSI has been busy with the other students question for T minutes, let q be the conditional probability that the problem is difficult. Find the value of q .

(b.) Conditioned on the information above, find the expected amount of time you have to wait from the time you arrive until the other students question is answered.

(c.) Now suppose two GSI's share a room and the professor is holding office hours in a different room. Both GSI's in the shared room are busy helping a student, and each has been answering questions for T minutes (there are no other students in the room). The amount of time the professor takes to answer a question is exponentially distributed with rate $\lambda = 6$ regardless of the difficulty. Supposing that the professor's room has two students (one of whom is being helped), in which room should you ask your question?

4. **Drawing Batteries I**

You have an endless box of used batteries. Assume that the number of hours remaining in a battery is i.i.d., uniformly distributed on $[0, 1]$.

(a.) Suppose you draw n batteries. Suppose that the ith battery you draw has X_i hours remaining. What is $P(X_1 \leq X_2 \leq \cdots \leq X_n)$?

Now, you draw batteries until you have enough batteries to last one hour. Let N be the number of batteries you draw.

(b.) What is $P(X_1 + X_2 \leq 1)$? What about $P(X_1 + X_2 + X_3 \leq 1)$?

(c.) Find the distribution and expectation of N.

Hint: Try to relate $P(X_1 + X_2 + \cdots + X_N \leq 1)$ to the quantity you found in part a.

5. **Finite Population Correction**

Consider a model of sampling in which we randomly draw a sample of n people, without replacement, from a population of N members. We are interested in, say, the height of the population. Assume that an individual's height is distributed with mean μ and variance σ^2. Let X_i denote the height of the ith individual in our sample, $i = 1, \ldots, n$. Let x_k, $k = 1, \ldots, m$ denote the possible values for X_i. (Since the population is finite, there are only finitely many possible values for X_i.)

(a) Calculate $E[X_j]$.

(b) Prove that for random variables $X_1, \ldots, X_N$,

$$\operatorname{var}(X_1 + \cdots + X_N) = \sum_{i=1}^{N} \operatorname{var}(X_i) + \sum_{i \neq j} \operatorname{cov}(X_i, X_j),$$

where the second summation ranges over all $(i, j) \in \{1, \ldots, N\}^2$ such that $i \neq j$. Do not assume that the random variables are independent.

(c) Calculate $\text{cov}(X_i, X_j)$ for $i \neq j$.

(d) Using the result you just calculated, calculate $\text{var}(\bar{X})$, where

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

is the sample mean. What is $\text{var}(\bar{X})/s^2$, where $s^2 = \sigma^2/n$ is the variance when sampling with replacement? (This is known as the **finite population correction**. Let $N \to \infty$ to see why an "infinite population" does not suffer from the same problem.)

6. **Auction Theory**

This problem explores auction theory and is meant to be done at the same time as the lab.

In auction theory, n bidders have **valuations** which represent how much they value an item; we will make the simplifying assumption that the valuations are i.i.d. with density $f(x)$. In the first-price auction, the bidder who makes the highest bid wins the item and pays his/her bid. In the second-price auction, the bidder who makes the highest bid wins the auction, and pays an amount equal to the *second-highest* bid. A strategy for the auction is a **bidding function** $\beta(x)$, where x is the bidder's valuation. The bidding function determines how much to bid as a function of the bidder's valuation, and the goal is to find a bidding function $\beta(\cdot)$ which maximizes your expected utility (0 if you do not win, and your valuation minus the amount of money you bid if you do win).

(a) For the first-price auction, consider the following scenario: each person draws his/her valuation uniformly from the interval $(0, 1)$ (so $f(x) = 1$ for $x \in (0, 1)$). Suppose that the other bidders bid their own valuations (they use $\beta(x) = x$, the identity bidding function). Consider the case where there is only one other bidder. The Donald insists that you should make a 'yuge' bid and always bid $\beta(x) = 1$. Your friend Bernie tells The Donald that it would be better to bid $\beta(x) = \frac{x}{2}$. Who is correct?

(b) Consider the same situation as the previous part, but now assume that there are n other bidders. The Donald again suggests that $\beta(x) = 1$ is a great bid, a fantastic bid, the best bid. Your friend Bernie suggests $\beta(x) = \frac{n}{n+1}x$. Who is correct this time?

(c) Consider a second-price auction where the bidders' valuations are i.i.d. with the exponential density (with parameter λ). Again, they use the identity bidding function, $\beta(x) = x$. What is the distribution of the price P at which the item sells?

7. **Auctions: Bayesian Nash Equilibrium (Optional: This problem is of a more theoretical nature and will not be tested on the exam)**

A **Bayesian Nash equilibrium** is a strategy for each player, such that no player has an incentive to change strategies. In other words, no player can improve his/her expected utility by changing his/her strategy. *The contents* *of this question will not be tested, but this question is provided as a way for you to further explore auction theory if you are interested.*

In this question, we will derive the Bayesian Nash equilibrium for the first-price auction, under the assumption that the valuations are i.i.d. with common density function $f(x)$. By symmetry, in the Bayesian Nash equilibrium, each bidder should use the same bid function $\beta(\cdot)$. We further assume that $\beta(\cdot)$ is differentiable and strictly increasing.

(a) Suppose that your valuation is x. Let X_i denote the valuation of player i, $i = 1, \ldots, n-1$ (your own valuation is known to you as a fixed real number, whereas the valuations of other players are modeled as random variables whose value is unknown). What is your expected utility when you bid b, assuming that the other $n-1$ bidders bid according to $\beta(\cdot)$? Write your answers in terms of the CDF $F(x) := \int_{-\infty}^{x} f(x)\,\mathrm{d}x$.

(b) Differentiate the expression you obtained with respect to b. *Hint*: You may need to use the Inverse Function Theorem, which states that the derivative of an inverse function is given by

$$\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.$$

(c) Now, suppose that you bid according to $\beta(\cdot)$ as well, i.e. $b = \beta(x)$. Under this assumption, set the result from the previous part to 0 and solve for $\beta(x)$.

(d) For the second-price auction, suppose that the other $n-1$ bidders bid their own valuations, i.e. they use the identity bidding function $\beta(x) = x$. Prove that it is optimal for you to bid your own valuation. The strategy of bidding your own valuation in the second-price auction is thus a Bayesian Nash equilibrium.