UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 10
Spring 2018

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1. **Random Graph**

   Consider a random undirected graph on n vertices, where each of the $\binom{n}{2}$ possible edges is present with probality p independently of all the other edges. If $p = 0$ we have a fully empty graph with n completely disconnected vertices; in contrast, if $p = 1$, every edge exists, the network is an n-clique, and every vertex is a distance one from every other vertex.

   (a) Fix a particular vertex of the graph, and let D be a random variable which is equal to the degree of this vertex. What is the PMF of D? Calculate $\lambda \triangleq \mathbb{E}[D]$.

   (b) Assume that $c = np$ is a constant, independent of n. For large values of n, how you would approximate the PMF of D?

2. **Isolated Vertices**

   Consider a Erdös-Renyi random graph $\mathcal{G}(n, p(n))$, where n is the number of vertices and $p(n)$ is the probability that a specific edge appears in the graph. Let X_n be the number of isolated vertices in $\mathcal{G}(n, p(n))$. Show that

$$\mathbb{E}[X_n] \xrightarrow{n\to\infty} \begin{cases} \infty, & p(n) \ll \dfrac{\ln n}{n}, \\ \exp(-c), & p(n) = \dfrac{\ln n + c}{n}, \\ 0, & p(n) \gg \dfrac{\ln n}{n}, \end{cases}$$

where the notation $p(n) \ll f(n)$ means that $p(n)/f(n) \to 0$ as $n \to \infty$, and $p(n) \gg f(n)$ means $p(n)/f(n) \to \infty$ as $n \to \infty$. Show also that in the third case, $p(n) \gg (\ln n)/n$, we have $X_n \to 0$ in probability as well.

3. Sub-Critical Forest

Consider a random graph $\mathcal{G}(n, p(n))$ where $p(n) \ll 1/n$ (this is called the **sub-critical phase**). Show that the probability that $\mathcal{G}(n, p(n))$ is a forest, i.e. contains no cycles, tends to 1 as $n \to \infty$. [If X_n is the number of cycles, compute $\mathbb{E}[X_n]$ and show that $\mathbb{E}[X_n] \to 0$. Then, apply the First Moment Method.]