

Midterm 1 Review
Spring 2018

1. Expected Distance

Let a_1, a_2, \dots, a_n be a random permutation of $\{1, 2, \dots, n\}$, equally likely to be any of the $n!$ possible permutations. When sorting the list a_1, a_2, \dots, a_n , the element a_i must move a distance of $|a_i - i|$ places from its current position to reach the position in the sorted order. Find

$$\mathbb{E} \left[\sum_{i=1}^n |a_i - i| \right],$$

the expected total distance that the elements will have to be moved.

Note: You can use the formula

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

in order to simplify your answer.

2. Exponential Queue

You have just finished grocery shopping and now you line up to purchase your items. You are the currently only person in line. There are two cashiers, and they are both serving a customer. The service time of the cashiers are independent random variables. Assume that you go to the first available cashier.

- Suppose that the first cashier is slower, and his service time is exponentially distributed with density $f_1(x) = \lambda_1 e^{-\lambda_1 x} \mathbb{1}\{x \geq 0\}$. The second cashier is slightly faster, and her service time is exponentially distributed with density $f_2(x) = \lambda_2 e^{-\lambda_2 x} \mathbb{1}\{x \geq 0\}$. What is the probability p_1 that the first cashier finishes before the second cashier?
- Out of the three people waiting for service (the two people currently being served and you), what is the probability that you are the last person to leave? (You may leave your answer in terms of the p_1 , the answer to the previous part.)
- Let the service times of the two cashiers be X_1 and X_2 respectively. What is $\mathbb{E}[Y | Z]$, where $Y = \max(X_1, X_2)$ and $Z = \min(X_1, X_2)$? Again, you may leave your answers in terms of p_1 . (*Hint:* Think about the Memoryless Property.)

(d) Compute the joint density of X_1 and $X_1 + X_2$.

3. Combining Transforms

Let X , Y , and Z be independent random variables. X is Bernoulli with $p = 1/4$. Y is exponential with parameter 3. Z is Poisson with parameter 5.

- (a) Find the transform of $5Z + 1$.
- (b) Find the transform of $X + Y$.
- (c) Consider the new random variable $U = XY + (1 - X)Z$. Find the transform associated with U .

4. Tricky Markov Bound

Suppose $\mathbb{E}[X] = 0$, $\text{var}(X) = \sigma^2 < \infty$, and $\alpha > 0$. Prove the following bound:

$$\mathbb{P}(X \geq \alpha) \leq \frac{\sigma^2}{\alpha^2 + \sigma^2}.$$

5. Chernoff Bound Application: Load Balancing

Here, we will give an application for the Chernoff bound which is instrumental for calculating confidence intervals. However, we will need a slightly more general version of the bound that works for any Bernoulli random variables. For any positive integer n , if X_1, \dots, X_n are i.i.d. Bernoulli, with $\mathbb{P}(X_i = 1) = p$, and $S_n = \sum_{i=1}^n X_i$, then the following bound holds for $0 \leq \varepsilon \leq 1$:

$$\mathbb{P}(S_n > (1 + \varepsilon)np) \leq \exp\left(-\frac{\varepsilon^2 np}{3}\right). \quad (1)$$

You may take (1) as a fact (or try to prove it on your own if you want!).

Here is the setting: there are k (k a positive integer) servers and n users. The simplest load balancing scheme is simply to assign each user to a server chosen uniformly at random (think of the users as “balls” and we are tossing them into server “bins”). By using the union bound, show that with probability at least $1 - 1/k^2$, the maximum load of any server is at most $n/k + 3\sqrt{\ln k} \sqrt{n/k}$. (You may assume that n is much larger than k .)