Electrical Engineering 126: Probability & Random Processes

Final Cheat Sheet

Spring 2019

1 Distributions

- $X \sim \text{Bernoulli}(p)$, $p \in [0, 1]$.
  PMF: $p_X(x) = p^x (1-p)^{1-x}$, $x \in \{0, 1\}$.
  MGF: $M_X(s) = 1 - p + p \exp s$.
  Moments: $\mathbb{E}[X] = p$, $\operatorname{var} X = p(1-p)$.
- $X \sim \text{Binomial}(n, p)$, $n \in \mathbb{Z}_+$, $p \in [0, 1]$.
  PMF: $p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$, $x \in \{0, \dots, n\}$.
  MGF: $M_X(s) = (1 - p + p \exp s)^n$.
  Moments: $\mathbb{E}[X] = np$, $\operatorname{var} X = np(1-p)$.
- $X \sim \text{Geometric}(p)$, $p \in (0, 1)$.
  PMF: $p_X(x) = pq^{x-1}$, $x \in \mathbb{Z}_+$, $q = 1 - p$.
  MGF: $M_X(s) = (p \exp s)/(1 - q \exp s)$, $s < \ln(1/q)$.
  Moments: $\mathbb{E}[X] = p^{-1}$, $\operatorname{var} X = q/p^2$.
- $X \sim \text{Poisson}(\lambda)$, $\lambda > 0$.
  PMF: $p_X(x) = \lambda^x \exp(-\lambda)/x!$, $x \in \mathbb{N}$.
  MGF: $M_X(s) = \exp(\lambda(\exp s - 1))$.
  Moments: $\mathbb{E}[X] = \lambda$, $\operatorname{var} X = \lambda$.
  X, Y independent, $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu) \implies X + Y \sim \text{Poisson}(\lambda + \mu)$.
- $X \sim \text{Uniform}[a, b]$, $a < b$.
  PDF: $f_X(x) = (b-a)^{-1}$, $x \in [a, b]$.
  MGF: $M_X(s) = (\exp(sb) - \exp(sa))/(s(b-a))$.
  Moments: $\mathbb{E}[X] = (a+b)/2$, $\operatorname{var} X = (b-a)^2/12$.
- $X \sim \text{Exponential}(\lambda)$, $\lambda > 0$.
  PDF: $f_X(x) = \lambda \exp(-\lambda x)$, $x > 0$.
  CDF: $F_X(x) = (1 - \exp(-\lambda x)) \mathbb{1}_{\{x \ge 0\}}$.
  MGF: $M_X(s) = \lambda/(\lambda - s)$, $s < \lambda$.
  Moments: $\mathbb{E}[X] = \lambda^{-1}$, $\operatorname{var} X = \lambda^{-2}$.
- $X \sim \mathcal{N}(\mu, \sigma^2)$, $\mu \in \mathbb{R}$, $\sigma^2 > 0$.
  PDF: $f_X(x) = (\sqrt{2\pi}\sigma)^{-1} \exp\{-(x-\mu)^2/(2\sigma^2)\}$.
  CDF: $F_X(x) = \Phi(x)$.
  MGF: $M_X(s) = \exp(\mu s + \sigma^2 s^2/2)$.
  Moments: $\mathbb{E}[X] = \mu$, $\operatorname{var} X = \sigma^2$.
  Gaussian Q-Function: $Q(x) = P(X > x\sigma + \mu) \implies F_X(x) = Q(\frac{x-\mu}{\sigma})$
  X, Y independent, $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $Y \sim \mathcal{N}(\mu_2, \sigma_2^2) \implies X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Continued:

- $X \sim \text{Pascal}(k, p)$, $k \in \mathbb{Z}_+$, $p \in (0, 1)$.
  Sum of k i.i.d. Geometric(p).
  PMF: $p_X(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}$, $x = k, k+1, k+2, \dots$.
- $X \sim \text{Erlang}(k, \lambda)$, $k \in \mathbb{Z}_+$, $\lambda > 0$.
  Sum of k i.i.d. Exponential(λ).
  PDF: $f_X(x) = \lambda^k x^{k-1} \exp(-\lambda x)/(k-1)!$, $x \ge 0$.
- $X \sim \mathcal{N}_n(\mu, \Sigma)$, $n \in \mathbb{Z}_+$ (joint Gaussian).
  PDF (assuming Σ invertible): For $x \in \mathbb{R}^n$,
  $f_X(x) = [(2\pi)^n \det \Sigma]^{-1/2} \exp\{-(x-\mu)^{\mathsf{T}} \Sigma^{-1} (x-\mu)/2\}$.
  MGF: $M_X(s) = \mathbb{E}[\exp(s^{\mathsf{T}} X)] = \exp(\mu^{\mathsf{T}} s + s^{\mathsf{T}} \Sigma s/2)$.
  Moments: $\mathbb{E}[X] = \mu$, $\operatorname{cov} X = \Sigma$.

2 Definitions & Equations

<u>Tail Sum</u>: For $X \ge 0$, $\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X \ge x)\,\mathrm{d}x$.

<u>Variance</u>: $\operatorname{var} X = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$. <u>Sum</u>: $\operatorname{var} \sum_{i=1}^n X_i = \sum_{i=1}^n \operatorname{var} X_i + \sum_{i \ne j} \operatorname{cov}(X_i, X_j)$.

<u>Covariance</u>: $\operatorname{cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$. <u>Matrix</u>: If $X = (X_1, \dots, X_n)$, $(\operatorname{cov} X)_{i,j} = \operatorname{cov}(X_i, X_j)$.

<u>Correlation</u>: $\rho(X, Y) = \operatorname{cov}(X, Y)/\sqrt{(\operatorname{var} X)(\operatorname{var} Y)}$.

<u>Entropy</u>: $H(X) = -\sum_{x \in \mathcal{X}} p(x) \log_2 p(x) = -\mathbb{E}[\log_2 p(X)]$.

<u>Order Statistics</u>: $f_{X^{(i)}}(x) = n\binom{n-1}{i-1} f(x) F(x)^{i-1} (1 - F(x))^{n-i}$.

<u>MGF</u>: $M_X(s) = \mathbb{E}[\exp(sX)]$.

<u>Markov</u>: For $X \ge 0$, $x > 0$, $\mathbb{P}(X \ge x) \le \mathbb{E}[X]/x$.

<u>Chebyshev</u>: For $x > 0$, $\mathbb{P}(|X - \mathbb{E}[X]| \ge x) \le (\operatorname{var} X)/x^2$.

<u>Chernoff</u>: For all x, $\mathbb{P}(X \ge x) \le (M_X(s))/e^{sx}$ for all $s > 0$ where the MGF is defined.

<u>Geometric Series</u>: $\sum_{k=0}^{n-1} ar^k = \frac{a(1-r^n)}{1-r}, r \ne 1$

<u>LLSE</u>: $L[X \mid Y] - \mathbb{E}[X] = [\operatorname{cov}(X, Y)/(\operatorname{var} Y)](Y - \mathbb{E}[Y])$.

Kalman Dynamics: For $n \in \mathbb{N}$,

$$\begin{aligned} X_{n+1} &= AX_n + V_n, \\ Y_n &= CX_n + W_n, \end{aligned}$$

where $X_0, (V_n,\ n \in \mathbb{N}), (W_n,\ n \in \mathbb{N})$ are zero mean, orthogonal, with $\operatorname{cov} V_n = \Sigma_V$, $\operatorname{cov} W_n = \Sigma_W$ for all $n \in \mathbb{N}$.

Kalman Filter: Assume Σ_W is invertible.

$$\begin{aligned} \hat{X}_{n|n} &:= L[X_n \mid Y_0, Y_1, \dots, Y_n] = \hat{X}_{n|n-1} + K_n \tilde{Y}_n, \\ \hat{X}_{n|n-1} &:= L[X_n \mid Y_0, Y_1, \dots, Y_{n-1}] = A\hat{X}_{n-1|n-1}, \\ \tilde{Y}_n &:= Y_n - L[Y_n \mid Y_0, Y_1, \dots, Y_{n-1}] = Y_n - C\hat{X}_{n|n-1}, \\ K_n &= \Sigma_{n|n-1} C^{\mathsf{T}} (C\Sigma_{n|n-1} C^{\mathsf{T}} + \Sigma_W)^{-1}, \\ \Sigma_{n|n-1} &:= \operatorname{cov}(X_n - \hat{X}_{n|n-1}) = A\Sigma_{n-1|n-1} A^{\mathsf{T}} + \Sigma_V, \\ \Sigma_{n|n} &:= \operatorname{cov}(X_n - \hat{X}_{n|n}) = (I - K_n C)\Sigma_{n|n-1}. \end{aligned}$$