EECS 126 Probability and Random Processes University of California, Berkeley: Spring 2017
(**Solutions**)
Kannan Ramchandran May 10, 2017

Final Exam

Last name	First name	SID

Name of student on your left:
Name of student on your right:

- DO NOT open the exam until instructed to do so.
- The total number of points is **110**, but a score of $\geq$ **100** is considered perfect.
- You have 10 minutes to read this exam without writing anything and 150 minutes to work on the problems.
- Box your final answers.
- **Remember to write your name and SID on the top left corner of every sheet of paper.**
- **Do not write on the reverse sides of the pages.**
- All eletronic devices must be turned off. Textbooks, computers, calculators, etc. are prohibited.
- No form of collaboration between students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.
- **You must include explanations to receive credit.**

Problem	Max	Points	Problem	Max	Points
1	12		7	12	
2	24		8	10	
3	12		9	8	
4	10		10	1	
5	10				
6	12				
Total				110	

Problem 1. (12 pts, 3 points each) You must give brief explanations in the provided boxes to get any credit.

(a) If two random variables X and Y are uncorrelated and independent, then they are jointly Gaussian.

True or False: False
Explanation:

(b) The following statement holds for any random variables X and Y:

$$E[(X - E[X|Y])(\cos Y)] = 0$$

True or False: True
Explanation:

(c) Consider the sequence X_n where $X_0 = 0, X_1 = 1$ and the dynamics are given by:

$$X_{n+1} = \begin{cases} X_n + X_{n-1} & \text{w.p. } \frac{1}{2} \\ |X_n - X_{n-1}| & \text{w.p. } \frac{1}{2} \end{cases}$$

The sequence $\{X_n\}$ is a Markov Chain.

True or False: False
Explanation:

(d) Consider a system with initial position X_0 and the following dynamics:

$$X_{n+1} = aX_n + V_n$$
$$Y_n = cX_n + W_n$$

where V_n and W_n are independent sources of noise. The Kalman filter can always be used to recover the MMSE of X_n given the observations $Y_1, Y_2, \ldots, Y_n$.

True or False: False
Explanation:

Problem 2. (24 pts, 6 pts each) Parts $(a), (b), (c)$ and (d) are short answer questions and unrelated to each other.

(a.) You would like to measure a random, zero-mean quantity X with known second moment $E[X^2] = 1$. However, you receive noisy measurements $Y = X + Z$, where Z is independent, zero-mean noise and $E[Z^2] = 1$. Find the LLSE of X given your noisy measurement and draw a vector space diagram illustrating your solution.

Solution: $\hat{X} = \frac{1}{2}Y$.

(b.) You would like to communicate to your friend Claude which one of $n = 200$ possible events (messages) occurred. Unfortunately, you are stuck using a channel which takes in 100 bits and randomly erases exactly 50 of these bits. To this end, you and Claude agree offline to use a dictionary designed as follows:

1. Flip 2000 fair coins (map a heads to 1 and tails to 0)
2. Let the codeword corresponding to message i be the result of coin flips $(i-1)100+1$ to $i(100)$, for $1 \leq i \leq 200$

You observe that message 5 has occurred and would like to convey that to Claude by sending the corresponding codeword over the channel. Claude is able to decode the codeword if he finds a *unique* match between the 50 bits he received and the first 50 bits in any of the codewords in the dictionary. Using the union bound, find an upper bound on the probability that Claude is unable to decode.

Solution: Let P_e be the probability of error. Let the codeword corresponding to the ith message be c_i. We note that $P(c_i = c_5) = 2^{-50}$. We thus have:

$$\begin{aligned} P_e &= P\left(\bigcup_{i \neq 5} 1_{c_i = c_5}\right) \\ &\leq \sum_{i \neq 5} P(c_i = c_5) \\ &= 199 \cdot 2^{-50} \end{aligned}$$

(c.) Let $Y_n = \min\{X_1, X_2, \ldots, X_n\}$, where X_i are iid and $X_i \sim U[0,1]$. Does Y_n converge in probability? If so, what does it converge to?

Solution: Yes, $Y_n \to 0$. To see this:

$$
\begin{aligned}
P(|Y_n| \geq \epsilon) &= P(\min\{X_1, \ldots, X_n\} \geq \epsilon) \\
&= P(X_1 \geq \epsilon)^n \qquad = (1-\epsilon)^n
\end{aligned}
$$

Since $(1-\epsilon) < 0$, $(1-\epsilon)^n \to 0$.

(d.) Let $X \sim \mathcal{N}(1,1)$ and $Y \sim \mathcal{N}(0,1)$ be jointly Gaussian with covariance $c = \frac{1}{2}$. What is $\Pr(X > Y)$?

Solution: We are interested in $P(X > Y) = P(X - Y > 0)$. Note that since X and Y are JG, $X - Y$ is Gaussian with mean $E[X] - E[Y] = 1$. Additionally, we see:

$$
\begin{aligned}
E[(X-Y)^2] - E[(X-Y)]^2 &= \mathrm{Var}(X) + \mathrm{Var}(Y) + \mathrm{cov}(X,Y) \\
&= 1
\end{aligned}
$$

Thus, $X - Y \sim \mathcal{N}(1,1)$. We thus are interested in:

$$
\begin{aligned}
P(X - Y > 0) &= P(\mathcal{N}(0,1) > -1) \\
&= P(\mathcal{N}(0,1)) < 1) \\
&= 0.8413
\end{aligned}
$$

Problem 3. (12 pts) Consider a random graph on n vertices in which each edge appears independently with probability p. Let E be the number of edges in the graph.

(a) (5 pts) Smart Alec claims that the maximum likelihood estimate of p given E is given by $\hat{p} = \frac{2E}{n(n-1)}$. Is Smart Alec correct?

Solution: Yes, Smart Alec is correct. To see this, note that:

$$P(E|p) = \binom{\binom{n}{2}}{E} p^E (1-p)^E$$

Taking the logarithm, differentiating and setting to 0, we can see that $p = \frac{2E}{n(n-1)}$, so Smart Alec is correct.

(b) (7 pts) Consider the case where n is getting large. Using the CLT, find a 95% confidence-level estimate for p Smart Alec's estimator $\hat{p}$ from the previous part. Your answer should not include p.

Solution: By the CLT, $E \sim \mathcal{N}(\binom{n}{2}p, \binom{n}{2}p(1-p))$, so that

$$\hat{p} \sim \mathcal{N}\left(p, \frac{2p(1-p)}{n(n-1)}\right).$$

We can upper bound $p(1-p) \leq 1/4$. An approximate 95% confidence interval is given by $\hat{p} \pm 2\sigma_{\hat{p}}$ (you could also give the slightly more accurate interval $\hat{p} \pm 1.96\sigma_{\hat{p}}$ if desired). So, our interval is $\hat{p} \pm \sqrt{2/(n(n-1))}$.

Problem 4. (10 pts) Consider a 3×3 chessboard. At time 0, the King is situated in the top left corner. At each time step, the King randomly selects a valid move and makes it. That is, at each time step, the King randomly selects an adjacent square (which can be diagonal) and moves to it.

(a.) (5 pts) Let the position of the King at time n be given by X_n. Find the long-term fraction of time the King spends in each square.

Solution: Note that the random walk on this 3×3 square is irreducible and thus the long-term fraction of time in each state is the stationary distribution. We let π_i be the stationary distribution at each state in the Markov Chain with π_1 in the top left corner, π_3 the top right corner, π_7 the bottom left corner, π_9 the bottom right corner, π_2 the top middle square, π_4 the left middle square, π_6 the right middle square, π_8 the bottom middle square, and π_5 in the middle. We note that by symmetry, $\pi_2 = \pi_4 = \pi_6 = \pi_8$ and $\pi_1 = \pi_3 = \pi_7 = \pi_9$. We have:

$$\begin{aligned}
\pi_1 &= \frac{1}{8}\pi_5 + \frac{1}{5}\pi_2 + \frac{1}{5}\pi_4 \\
\pi_2 &= \frac{1}{3}\pi_1 + \frac{1}{5}\pi_4 + \frac{1}{8}\pi_5 + \frac{1}{5}\pi_6 + \frac{1}{3}\pi_3 \\
\pi_5 &= \frac{1}{3}\pi_1 + \frac{1}{5}\pi_2 + \frac{1}{3}\pi_3 + \frac{1}{5}\pi_4 + \frac{1}{5}\pi_6 + \frac{1}{3}\pi_7 + \frac{1}{5}\pi_8 + \frac{1}{3}\pi_9 \\
\sum_{i=1}^{9} \pi_i &= 1
\end{aligned}$$

Solving gives $\pi_1 = \frac{3}{40}, \pi_2 = \frac{1}{8}, \pi_5 = \frac{1}{5}$.

(b.) (5 pts) What is the expected amount of time until the King returns to the top left corner?

Solution: Let T_i be the number of steps the King takes before returning to state 1 for the ith time. Note that by the weak law of large numbers:

$$\frac{T_1 + T_2 + \cdots + T_k}{k} \to E[T_1]$$

Additionally, the long-term fraction of time the King spends in state 1 is given by:

$$\frac{k}{T_1 + T_2 + \cdots + T_k}$$

Thus, we can see that $E[T_1] = \frac{1}{\pi_1} = \frac{40}{3}$.

It was not necessary to see this argument. If the equations are set up, full credit is given.

Problem 5. (10 pts) Particles are escaping a nuclear plant according to a Poisson process with rate 12 particles per second. Each particle that escapes is contained in one of three chambers and is equally likely to end up in any of the three chambers. Suppose that the second arrival to the first chamber is after 1 second, the second arrival to the second chamber is after 2 seconds, and the second arrival to the third chamber is after 3 seconds. Let X_i be the time of the first arrival to the ith chamber.

(a.) (5 pts) Find the joint distribution of (X_1, X_2, X_3).

Solution: By Poisson Splitting, $X_1 \sim U[0,1], X_2 \sim U[0,2], X_3 \sim U[0,3]$ and the three random variables are independent. Thus, their joint distribution is given by:

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = \frac{1}{2} \cdot \frac{1}{3} 1_{0 \le x_1 \le 1, 0 \le x_2 \le 2, 0 \le x_3 \le 3}$$

(b.) (5 pts) Find $E[X_1^3 + X_3^3 | 2X_1 + X_2 = 2]$.

Note that $E[X_3^3 | 2X_1 + X_2 = 2] = E[X_3^3] = \frac{81}{12}$. Additionally, $X_1 | 2X_1 + X_2 \sim U[0,1]$, so $E[X_1^3 | 2X_1 + X_2 = 2] = \frac{1}{4}$. Thus, we have:

$$E[X_1^3 + X_3^3 | 2X_1 + X_2 = 2] = 7$$

Problem 6. (12 pts)

(a.) (5 pts) Let $X_1, X_2, \ldots, X_n$ be iid Gaussian random variables with unknown mean μ and unit variance. Find the MLE of μ given the observations $\{x_i\}_{i=1}^n$.

Solution:

$$P(x_1, \ldots, x_n|\mu) = \frac{1}{(2\pi)^{n/2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}$$

$$\hat{\mu} := MLE[\mu|x_1, \ldots, x_n] = \arg\max_{\mu}\left\{-\sum_{i=1}^{n}(x_i-\mu)^2\right\} = \frac{1}{n}\sum_{i=1}^{n}x_i.$$

(b.) (7 pts) Now suppose you observe only one sample X_1. You would like to test the two hypotheses:

$$H_1: \ X_1 \sim \mathcal{N}(0,1)$$
$$H_0: \ X_1 \sim \text{Exp}(1)$$

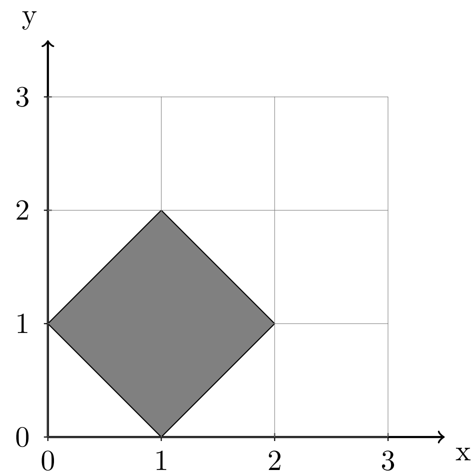
Formulate a hypothesis test to maximize the probability of correct decision subject to the probability of false alarm $\leq 1 - e^{-2}$.

Solution: First note that the likelihood is not monotonically decreasing or increasing. One can see that $L(y) = +\infty$ when $x < 0$, has a parabola shape between 0 and 2 and is monotonically decreasing between for $x > 2$. Thus, if the observed $x > 2$, one can simply set a threshold on the observed value rather than the likelihood function so that the threshold rule is $\hat{X} = 1$ if $x \leq \tau$. To find whether this is acceptable, we set:

$$\begin{aligned} P(\hat{X} = 1|H_0) &= P(x \leq \tau|H_0) \\ &= 1 - e^{-\tau} \end{aligned}$$

Setting this value equal to the PFA bound $1 - e^{-2}$, we can see that this corresponds to setting $\tau = 2$, and we are done.

Problem 7. (12 pts) Let X and Y have a uniform distribution on the region given in the Figure.



(a) (4 pts) Find the Moment-Generating Function (MGF) of X, $M_X(s) = E[e^{sX}]$.

Solution: Integrating the joint pdf over y, we obtain

$$f_X(x) = \begin{cases} x & 0 \le x \le 1, \\ 2-x & 1 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, X is identically distributed with $U_1 + U_2$ where U_1 and U_2 are iid uniform random variables on $[0,1]$. Then,

$$M_X(s) = (M_{U_1}(s))^2 = \left(\frac{e^s - 1}{s}\right)^2.$$

(b) (4 pts) Find the transform of $X+Y$, $M_{X+Y}(s)$.

Solution: Note that

$$F_{X+Y}(z) = P(X+Y \le z) = \begin{cases} 0 & z \le 1, \\ (z-1)/2 & 1 \le z \le 3, \\ 1 & z \ge 1. \end{cases}$$

Therefore, $X+Y$ is uniformly distributed on $[1,3]$, and

$$M_{X+Y}(s) = \int_1^3 \frac{1}{2} e^{sx} dx = \frac{e^{3s} - e^s}{2s}.$$

(c) (4 pts) Find $\mathrm{Var}(X+Y|X-Y \geq 0.5)$.

Solution: Considering the part the of the shaded area coinciding with the region $X-Y \geq 0$, we observe that $X+Y$ is still uniformly distributed on $[1,3]$; and therefore, $\mathrm{Var}(X+Y|X-Y \geq 0.5) = (3-1)^2/12 = 1/3$.

Problem 8. (10 pts) Consider a particle with initial position $X_0 \sim \text{Poi}(\lambda)$ and which moves according to the following dynamics:

$$\begin{aligned} X_{n+1} &= X_n + V_n \\ Y_n &= X_n + W_n \end{aligned}$$

where $V_n \sim \text{Poi}(\lambda)$ and $W_n \sim \text{Poi}(\lambda)$ are independent sources of noise.

(a.) (5 pts) Suppose that you observe only Y_3. Find the distribution and the MMSE of X_3 given this observation.

Solution: Here, we have $E[X_3|Y_3] = E[X_3|X_3+W_3]$. Note that one can view X_3 and W_3 as independent Poisson processes, W_3 having rate λ and X_3 having rate 4λ. Thus, the merged process has rate 5λ and $X_3|Y_3 \sim \text{Bin}(Y_3, \frac{4}{5})$ so that $E[X_3|Y_3] = \frac{4}{5}Y_3$.

(b.) (5 pts) Suppose that you see observations Y_0, Y_1. Find $L[X_1|Y_0, Y_1]$. That is, find the LLSE of X_1 given observations Y_0, Y_1.

Solution: Let $\tilde{X}_1 = X_1 - E[X_1], \tilde{Y}_0 = Y_0 - E[Y_0], \tilde{Y}_1 = Y_1 - E[Y_1]$. Then we have:

$$L[X_1|Y_0, Y_1] = E[X_1] + L[\tilde{X}_1|\tilde{Y}_0, \tilde{Y}_1]$$

Using the geometric view from the Kalman note, we can see that $L[\tilde{X}_1|\tilde{Y}_0, \tilde{Y}_1] = L[\tilde{X}_0|\tilde{Y}_0] + k_1(\tilde{Y}_1 - L[\tilde{Y}_1|\tilde{Y}_0]$. We can addtionally see that $k_1 = \frac{3}{5}$, and thus we have the estimate:

$$\begin{aligned} L[X_1|Y_0, Y_1] &= 2\lambda + \frac{3}{5}\tilde{Y}_1 + \frac{1}{5}\tilde{Y}_0 \\ &= \frac{3}{5}Y_1 + \frac{1}{5}Y_0 - \frac{\lambda}{5} \end{aligned}$$

Problem 9. (8 pts) Assume that the Markov chain $\{X_n, n \geq 0\}$ with states 0 and 1, and initial distribution $\pi_0(0) = \pi_0(1) = 0.5$ and $P(x, x') = 0.3$ for $x \neq x'$ and $P(x, x) = 0.7$ $(x, x' \in \{0, 1\})$. Assume also that X_n is observed through a BSC with error probability 0.1. The observations are denoted by Y_n. Suppose the observations are $(Y_0, \ldots, Y_4) = (0, 0, 1, 1, 1)$. Use the Viterbi algorithm to find the most likely sequence of the states $(X_0, \ldots, X_4)$. For this problem, you may use the following approximations: $\log 0.5 = -0.3, \log 0.1 = -1, \log 0.9 = -0.05, \log 0.3 = -0.523, \log 0.7 = -0.155$.

Solution: Running the Viterbi algorithm gives $(0, 0, 1, 1, 1)$.

Problem 10. (1 point) Please leave any feedback for the course staff here. What did you like and dislike about the course? What can we improve upon?

END OF THE EXAM.

Please check whether you have written your name and SID on every page.

Hope you enjoyed the class! You learned a lot!