UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: Probability and Random Processes

Homework I
Spring 2020

Problem 1. **Coin Flipping & Symmetry**
Alice and Bob have $2n+1$ fair coins (where $n \in \mathbb{Z}_{>0}$), each coin with probability of heads equal to $1/2$. Bob tosses $n+1$ coins, while Alice tosses the remaining n coins. Assuming independent coin tosses, show that the probability that, after all coins have been tossed, Bob will have gotten more heads than Alice is $1/2$.
Hint: This problem has a simple solution exploiting symmetry.

Problem 2. **Balls & Bins**
Let $n \in \mathbb{Z}_{>1}$. You throw n balls, one after the other, into n bins, so that each ball lands in one of the bins uniformly at random. What is an appropriate sample space to model this scenario? What is the probability that exactly one bin is empty?

Problem 3. **Secret Hitler**
In the game of Secret Hitler, there are 9 players. 4 of them are Fascist and 5 of them are Liberal. There is also a deck of 17 cards containing 11 Fascist "policies" and 6 Liberal "policies". Fascists want to play Fascist policies, and Liberals want to play Liberal policies. Here's how the play proceeds.

- A President and a Chancellor are chosen uniformly at random from the 9 players.
- The President draws 3 policies from the deck and gives 2 to the Chancellor.
- The Chancellor chooses one to play.

Now suppose you are the Chancellor, but the President gave you 2 Fascists. Being a Liberal, you wonder, did the President just happen to have 3 Fascist policies, or was the President a Fascist who secretly discarded a Liberal policy. In this scenario, what's the probability that the President is Fascist? Let's assume that Fascist presidents always try to discard Liberal policies.

Problem 4. **Passengers on a Plane**
There are N passengers in a plane with N assigned seats (N is a positive integer), but after boarding, the passengers take the seats randomly. Assuming all seating arrangements are equally likely, what is the probability that no passenger is in their assigned seat? Compute the probability when $N \to \infty$.
Hint: Use the inclusion-exclusion principle.

Problem 5. **Shooting free throws**
You shoot n free throws in a row, with your accuracy varying according to the following rule: the probability you make the ith shot is equal to the fraction of free throws you have made so far in the previous $i-1$ shots. E.g. if you have shot 5 free throws so far, and you made 3 of them, then your next shot has a 60% chance of going in.
Say you made the first free throw and missed the second free throw (Assume $n > 2$). For $1 \le x < n$, prove that the probability of making exactly x free throws is equal to $\frac{1}{n-1}$ (*Hint:* Use induction).

Problem 6. **Expanding the NBA**
The NBA is looking to expand to another city. In order to decide which city will receive a new team, the commissioner interviews potential owners from each of the N potential cities (N is a positive integer), one at a time. Unfortunately, the owners would like to know immediately after the interview whether their city will receive the team or not. The commissioner decides to use the following strategy: she will interview the first m owners and reject all of them ($m \in \{1, \dots, N\}$). After the mth owner is interviewed, she will pick the first city that is better than all previous cities. The cities are interviewed in a uniformly random order. What is the probability that the best city is selected? Assume that the commissioner has an objective method of scoring each city and that each city receives a unique score.
You should arrive at an exact answer for the probability in terms of a summation. Approximate your answer using $\sum_{i=1}^{n} i^{-1} \approx \ln n$ and find the optimal value of m that maximizes the probability that the best city is selected.

Problem 7. **[Optional] Deriving Facts from the Axioms**

1. Let $n \in \mathbb{Z}_{>0}$ and $A_1, \dots, A_n$ be any events. Prove the **union bound**: $\Pr(\bigcup_{i=1}^{n} A_i) \le \sum_{i=1}^{n} \Pr(A_i)$.
2. Let $A_1 \subseteq A_2 \subseteq \cdots$ be a sequence of increasing events. Prove that $\lim_{n\to\infty} \Pr(A_n) = \Pr(\bigcup_{i=1}^{\infty} A_i)$. [This can be viewed as a **continuity** property for probability measures.]
3. Let $A_1, A_2, \dots$ be a sequence of events. Prove that the union bound holds for countably many events: $\Pr(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} \Pr(A_i)$.

Problem 8. **[Optional] Tournament Probabilistic Proof**
In a *tournament* with n players (where n is a positive integer), each player plays against every other player for a total of $\binom{n}{2}$ games (assume that there are no ties). Let k be a positive integer. Is it always possible to find a tournament such that for any subset A of k players, there is a player who has beaten everyone in A? For such a tournament, let us say that every k-subset is dominated. For example, Figure 1 depicts the smallest tournament in which every 2-subset is dominated.
In fact, as long as $\binom{n}{k}(1-2^{-k})^{n-k} < 1$, it is possible to find a tournament of n players such that every k-subset is dominated. Prove this fact, and explain why it implies that for any positive integer k there exists a tournament such that every k-subset is dominated.

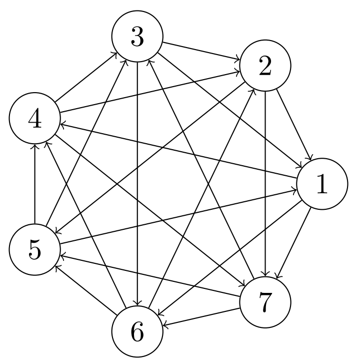

Figure 1: A tournament with 7 vertices such that every pair of players is beaten by a third player.