UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 2
Spring 2020

1. **Variance**

   If $X_1, \ldots, X_n$, where $n \in \mathbb{Z}_{>0}$, are i.i.d. random variables with zero-mean and unit variance, compute the variance of $(X_1 + \cdots + X_n)^2$. You may leave your answer in terms of $\mathbb{E}[X_1^4]$, which is assumed to be finite.

2. **Message Segmentation**

   The number of bytes N in a message has a geometric distribution with parameter p. Suppose that the message is segmented into packets, with each packet containing m bytes if possible, and any remaining bytes being put in the last packet. Let Q denote the number of full packets in the message, and let R denote the nubmer of bytes left over.

   (a) Find the joint PMF of Q and R. Pay attention on the support of the joint PMF.

   (b) Find the marginal PMFs of Q and R.

   (c) Repeat part (b), given that we know that $N > m$.

   Note: you can use the formulas

   $$\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a}, \text{ for } a \neq 1$$
   $$\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}, \text{ for } |x| < 1$$

   in order to simplify your answer.

3. **Packet Routing**

   Packets arriving at a switch are routed to either destination A (with probability p) or destination B (with probability $1-p$). The destination of each packet is chosen independently of each other. In the time interval $[0, 1]$, the number of arriving packets is Poisson(λ).

(a) Show that the number of packets routed to A is Poisson distributed. With what parameter?

(b) Are the number of packets routed to A and to B independent?

4. Poisson Recursion

Suppose X is Poisson distributed with parameter λ. Show that $\mathbb{E}(X^n) = \lambda\mathbb{E}[(X+1)^{n-1}]$. Use this to compute $\mathbb{E}(X^3)$.

5. Compact Arrays

Consider an array of n entries, where n is a positive integer. Each entry is chosen uniformly randomly from $\{0, \ldots, 9\}$. We want to make the array more compact, by putting all of the non-zero entries together at the front of the array. As an example, suppose we have the array

$$[6, 4, 0, 0, 5, 3, 0, 5, 1, 3].$$

After making the array compact, it now looks like

$$[6, 4, 5, 3, 5, 1, 3, 0, 0, 0].$$

Let i be a fixed positive integer in $\{1, \ldots, n\}$. Suppose that the ith entry of the array is non-zero (assume that the array is indexed starting from 1). Let X be a random variable which is equal to the index that the ith entry has been moved after making the array compact. Calculate $\mathbb{E}[X]$ and $\text{var}(X)$.

6. Almost fixed points of a permutation

Let Ω be the set of all permutations of the numbers $1, 2, ..., n$. Let an almost fixed point be defined as follows: If we put the numbers $i \in 1, 2, ..., n$ around a circle in clockwise order (such that 1 and n are next to each other) and then assign another number $\omega(i) \in 1, 2, ..., n$ to it, if the number $\omega(i)$ is next to i (or is equal to i), we will say that i is almost a fixed point. So, for the permutation $\omega(1) = 5, \omega(2) = 3, \omega(3) = 1, \omega(4) = 4, \omega(5) = 2$, we have that 1, 2, and 4 are almost fixed points.

Now, let $X(\omega)$ denote the number of almost fixed points in $\omega \in \Omega$. Find $\mathbb{E}[X]$ and $\text{var}(X)$. You may assume that $n \geq 5$.

7. [Bonus] Connected Random Graph

The bonus question is just for fun. You are not required to submit the bonus question, but do give it a try and write down your progress.

We start with the empty graph on n vertices, and iteratively we keep on adding undirected edges $\{u, v\}$ uniformly at random from the edges that are not so far present in the graph, until the graph is connected. Let X be a random variable which is equal to the total number of edges of the graph. Show that $\mathbb{E}[X] = O(n \log n)$.

Hint: consider the random variable X_k which is equal to the number of edges added while there are k connected components, until there are $k - 1$ connected components. Don't try to calculate $\mathbb{E}[X_k]$, an upper bound is enough.