UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 4
Spring 2020

1. **Transform Practice**

Consider a random variable Z with transform

$$M_Z(s) = \frac{a - 3s}{s^2 - 6s + 8}, \qquad \text{for } |s| < 2.$$

Calculate the following quantities:

(a) The numerical value of the parameter a.
(b) $\mathbb{E}[Z]$.
(c) $\text{var}(Z)$.

2. **Combining Transforms**

Let X, Y, and Z be independent random variables. X is Bernoulli with $p = 1/4$. Y is exponential with parameter 3. Z is Poisson with parameter 5.

(a) Find the transform of $5Z + 1$.
(b) Find the transform of $X + Y$.
(c) Consider the new random variable $U = XY + (1 - X)Z$. Find the transform associated with U.

3. **Gaussian Densities**

(a) Let $X_1 \sim \mathcal{N}(0, 1)$, $X_2 \sim \mathcal{N}(0, 1)$, where X_1 and X_2 are independent. Convolve the densities of X_1 and X_2 to show that $X_1 + X_2 \sim \mathcal{N}(0, 2)$.
(b) Let $X \sim \mathcal{N}(0, \sigma^2)$; find $\mathbb{E}[X^n]$ for $n \in \mathbb{N}$.
(c) Let $Z \sim \mathcal{N}(0, 1)$. For a random vector $(X_1, \ldots, X_n)$ where n is a positive integer and $X_1, \ldots, X_n$ are real-valued random variables, the expectation of $(X_1, \ldots, X_n)$ is the vector of elementwise expectations of each random variable and the **covariance matrix** of

$(X_1, \ldots, X_n)$ is the $n \times n$ matrix whose (i, j) entry is $\mathrm{cov}(X_i, X_j)$ for all $i, j \in \{1, \ldots, n\}$. Find the mean and covariance matrix of $(Z, \mathbf{1}\{Z > c\})$ in terms of ϕ and Φ, the standard Gaussian PDF and CDF respectively.

4. **A Chernoff Bound for the Sum of Coin Flips**

Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli(q) random variables with bias $q \in (0, 1)$, and call X their sum, $X = X_1 + \cdots + X_n$, which a Binomial(n, q) random variable, with mean $\mathbb{E}[X] = nq$.

(a) Let $\epsilon > 0$ such that $q + \epsilon < 1$, and define $p = q + \epsilon$. Show that for any $t > 0$,

$$\mathbb{P}(X \geq pn) \leq \exp\bigl(-n(tp - \ln \mathbb{E}[\mathrm{e}^{tX_1}])\bigr).$$

(b) The *Kullback-Leibler divergence* from the distribution Bernoulli(q) to the distribution Bernoulli(p), is defined as

$$D(p \parallel q) \triangleq p \ln \frac{p}{q} + (1 - p) \ln \frac{1 - p}{1 - q}.$$

The Kullback-Leibler divergence can be interpreted as a measure of how close the two distributions are. One motivation for this interpretation is that the Kullback-Leibler divergence is always nonnegative, i.e. $D(p \parallel q) \geq 0$, and $D(p \parallel q) = 0$ if and only if $p = q$. So it can be thought of as a 'distance' between the two Bernoulli distributions.

Optimize the previous bound over $t > 0$ and deduce that

$$\mathbb{P}(X \geq pn) \leq \mathrm{e}^{-nD(p \parallel q)}.$$

(c) Moreover, the Kullback-Leibler divergence is related to the square distance between the parameters p and q via the following inequality

$$D(p \parallel q) \geq 2(p - q)^2, \quad \text{for } p, q \in (0, 1).$$

Use this inequality in order to deduce that

$$\mathbb{P}\bigl(X \geq (q + \epsilon)n\bigr) \leq \mathrm{e}^{-2n\epsilon^2},$$

and

$$\mathbb{P}\bigl(X \leq (q - \epsilon)n\bigr) \leq \mathrm{e}^{-2n\epsilon^2}.$$

Hint: For the second bound use symmetry in order to avoid doing all the work again.

(d) Conclude that

$$\mathbb{P}(|X - qn| \geq \epsilon n) \leq 2\mathrm{e}^{-2n\epsilon^2}.$$