UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: PROBABILITY AND RANDOM PROCESSES

Homework VII
Spring 2020

Problem 1. **Markov Chain Practice**
Consider a Markov chain with three states 0, 1, and 2. The transition probabilities are $P(0,1) = P(0,2) = 1/2$, $P(1,0) = P(1,1) = 1/2$, and $P(2,0) = 2/3$, $P(2,2) = 1/3$.

(a) Classify the states in the chain. Is this chain periodic or aperiodic?

(b) In the long run, what fraction of time does the chain spend in state 1?

(c) Suppose that X_0 is chosen according to the steady state distribution. What is $\Pr(X_0 = 0 \mid X_2 = 2)$?

Problem 2. **Flea on a Triangle**
A flea hops about at random on the vertices of a triangle, with all jumps equally likely. Find the probability that after n ($n \in \mathbb{N}$) hops the flea is back where it started.

Problem 3. **Compression of a Markov Chain**
Consider an irreducible Markov chain $(X_n)_{n\in\mathbb{N}}$ as shown below.

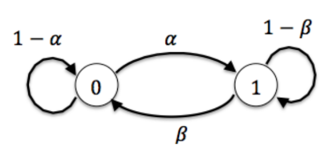

Figure 1: Two State Markov chain

Suppose $\alpha = \beta = p$ and $X_0 \sim B(\frac{1}{2})$. Roughly how many bits are needed to represent $(X_0, X_1, \dots, X_n)$?

Problem 4. **Random Walk on the Cube**
Consider the symmetric random walk on the vertices of the 3-dimensional unit cube where two vertices are connected by an edge if and only if the line connecting them is an edge of the cube. In other words, this is the random walk on the graph with 8 nodes each written as a string of 3 bits, so that the vertex set is $\{0,1\}^3$, and where two vertices are connected by an edge if and only if their corresponding bit strings differ in exactly one location.
This random walk is modified so that the nodes 000 and 111 are made absorbing.

(a) What are the communicating classes of the resulting Markov chain? For each class, determine its period, and whether it is transient or recurrent.

(b) For each transient state, what is the probability that the modified random walk started at that state gets absorbed in the state 000?

Problem 5. **Fly on a Graph**
A fly wanders around on a graph G with vertices $V = \{1, \ldots, 5\}$, shown in Figure 2.

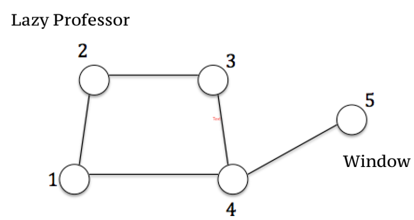

Figure 2: A fly wanders randomly on a graph.

(a) Suppose that the fly wanders as follows: if it is at node i at time n, then it chooses one of its neighbors j of i uniformly at random, and then wanders to node j at time $n+1$. For times $n = 0, 1, 2, \ldots$, let X_n be the fly's position at time n. Argue that $\{X_n,\ n \in \mathbb{N}\}$ is a Markov chain, and find the invariant distribution.

(b) Now for the process in part (a), suppose that the (not-to-be-named) professor sits at node 2 reading a heavy book. The professor is very lazy, so they don't move at all, but will drop the book on the fly if it reaches node 2 (killing it instantly). On the other hand, node 5 is a window that lets the fly escape. What is the probability that the fly escapes through the window supposing that it starts at node 1?

(c) Now suppose that the fly wanders as follows: when it is at node i at time n, it chooses uniformly from all neighbors of node i except for the one that it just came from. For times $n = 0, 1, 2, \ldots$, let Y_n be the fly's position at time n. Is this new process $\{Y_n, n \in \mathbb{N}\}$ a Markov chain? If it is, write down the probability transition matrix; if not, explain why it does not satisfy the definition of Markov chains.

Problem 6. **Choosing Two Good Movies**
You have a database of a countably infinite number of movies. Each movie has a rating that is uniformly distributed in $\{0, 1, 2, 3, 4, 5\}$ and you want to find two movies such that the sum of their rating is greater than 7.5. Assume that you choose movies from the database one by one and keep the movie with the highest rating so far. You stop when you find that the sum of the ratings of the last movie you have chosen and the movie with the highest rating among all the previous movies is greater than 7.5.

1. Define an appropriate Markov chain and use the first step equations in order to find the expected number of movies you will have to choose.

2. Now assume that the ratings of the movies are uniformly distributed in the interval $[0, 5]$. Write the first step equations for the expected number of movies you will have to choose in this case.

Problem 7. **[Optional] Two-State Chain with Linear Algebra**
Consider the Markov chain $(X_n,\ n \in \mathbb{N})$, shown in Figure 3, where $\alpha, \beta \in (0, 1)$.

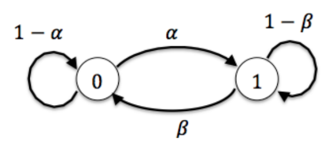

Figure 3: Two State Markov chain

1. Find the probability transition matrix P.

2. Find two real numbers λ_1 and λ_2 such that there exists two non-zero vectors u_1 and u_2 such that $Pu_i = \lambda_i u_i$ for $i = 1, 2$. Further, show that P can be written as $P = U\Lambda U^{-1}$, where U and Λ are 2×2 matrices and Λ is a diagonal matrix.

   Hint: This is called the eigendecomposition of a matrix.

3. Find P^n in terms of U and Λ for each $n \in \mathbb{N}$.

4. Assume that $X_0 = 0$. Use the result in part (c) to compute the PMF of X_n for all $n \in \mathbb{N}$.

5. What does the fraction of time spent in state 0, $n^{-1} \sum_{i=1}^{n} \mathbf{1}\{X_i = 0\}$, converge to (almost surely) as $n \to \infty$?