UC Berkeley
Department of Electrical Engineering and Computer Sciences

EE126: PROBABILITY AND RANDOM PROCESSES

Homework IX
Spring 2020

Problem 1. **Poisson Process Warm-Up**
Consider a Poisson process $\{N_t, t \geq 0\}$ with rate $\lambda = 1$. Let random variable S_i denote the time of the ith arrival, where i is a positive integer.

(a) Given $S_3 = s$, where $s > 0$, find the joint distribution of S_1 and S_2.

(b) Find $\mathbb{E}[S_2 \mid S_3 = s]$.

(c) Find $\mathbb{E}[S_3 \mid N_1 = 2]$.

(d) Give an interpretation, in terms of a Poisson process with rate λ, of the following fact:

If N is a geometric random variable with parameter p, and $(X_i)_{i\in\mathbb{N}}$ are i.i.d. exponential random variables with parameter λ, then $X_1 + \cdots + X_N$ has the exponential distribution with parameter λp.

Problem 2. **Bus Arrivals at Cory Hall**
Starting at time 0, the 52 line makes stops at Cory Hall according to a Poisson process of rate λ. Students arrive at the stop according to an independent Poisson process of rate μ. Every time the bus arrives, all students waiting get on.

1. Given that the interarrival time between bus $i-1$ and bus i is x, find the distribution for the number of students entering the ith bus. (Here, x is a given number, not a random quantity.)

2. Given that a bus arrived at 9:30 AM, find the distribution for the number of students that will get on the next bus.

3. Find the distribution of the number of students getting on the next bus to arrive after 9:30 AM, assuming that time 0 was infinitely far in the past.

Problem 3. **Integrated Shot Noise**
A noise impulse occurs at time $t = 0$, and later impulses occur at Poisson process times with mean rate $\lambda > 0$. Each impulse instantaneously charges a capacitor to 1 volt, and the voltage then decreases exponentially as e^{-t} until the next impulse occurs. Let V_t denote the voltage at time t. Let

$$Z_n = \int_0^{T_n} V_t \, dt$$

be the integrated voltage up to the time of the n^{th} impulse occurring after $t = 0$, for each positive integer n.
Find $\mathbb{E}[Z_n]$ and $var Z_n$.

Problem 4. **System Shocks**
For a positive integer n, let $X_1, \ldots, X_n$ be independent exponentially distributed random variables, each with mean 1. Let $\gamma > 0$.
A system experiences shocks at times $k = 1, \ldots, n$. The size of the shock at time k is X_k.

1. Suppose that the system fails if any shock exceeds the value γ. What is the probability of system failure?

2. Suppose instead that the effect of the shocks is cumulative, i.e., the system fails when the total amount of shock received exceeds γ. What is the probability of system failure?

Problem 5. **Random Telegraph Wave**
Let $\{N_t,\ t \geq 0\}$ be a Poisson process with rate λ and define $X_t = X_0(-1)^{N_t}$ where $X_0 \in \{0, 1\}$ is a random variable independent of N_t.

(a) Does the process X_t have independent increments?

(b) Calculate $\Pr(X_t = 1)$ if $\Pr(X_0 = 1) = p$.

(c) Assume that $p = 0.5$. Calculate $\mathbb{E}[X_{t+s}X_s]$ for $s, t \geq 0$.

Problem 6. **Sum-Quota Sampling**
Consider the problem of estimating the mean interarrival time of a Poisson process. In what follows, recall that N_t denotes the number of arrivals by time t, where $t \geq 0$. *Sum-quota sampling* is a form of sampling in which the number of samples is not fixed in advance; instead, we wait until a fixed *time* t, and take the average of the interarrival times seen so far. If we let X_i denote the ith interarrival time, then

$$\bar{X} = \frac{X_1 + \cdots + X_{N_t}}{N_t}.$$

Of course, the above quantity is not defined when $N_t = 0$, so instead we must condition on the event $\{N_t > 0\}$. Compute $\mathbb{E}[\bar{X} \mid N_t > 0]$, assuming that $(N_t,\ t \geq 0)$ is a Poisson process of rate λ.