UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Discussion 12
Spring 2021

1. **Hypothesis Testing for Bernoulli Random Variables**

Assume that

- If $X = 0$, then $Y \sim \text{Bernoulli}(1/4)$.
- If $X = 1$, then $Y \sim \text{Bernoulli}(3/4)$.

Using the Neyman-Pearson formulation of hypothesis testing, find the optimal randomized *decision rule* $r : \{0,1\} \to \{0,1\}$ with respect to the criterion

$$\min_{\text{randomized } r:\{0,1\}\to\{0,1\}} \Pr\big(r(Y) = 0 \mid X = 1\big)$$
$$\text{s.t. } \Pr\big(r(Y) = 1 \mid X = 0\big) \le \beta,$$

where $\beta \in [0,1]$ is a given upper bound on the false positive probability.

2. **MAP Hypothesis Testing**

Let X have prior probabilities $\Pr(X = 0) = \pi_0, \Pr(X = 1) = \pi_1$, and Y be the observed variable.

(a) Recall that the MAP (binary) hypothesis test says:

$$\hat{X}_{MAP}(y) = \arg\max_x p_{X|Y}(x|y) = \begin{cases} 1 & \text{if } \frac{p_{Y|X}(y|1)}{p_{Y|X}(y|0)} \ge \frac{\pi_0}{\pi_1} \\ 0 & \text{else} \end{cases}$$

Show that the MAP test $\hat{X}_{MAP}$ minimizes the probability of error $P(\hat{X}(Y) \neq X)$ over all tests $\hat{X}$.

(b) Suppose instead of minimizing the expected probability of error, we want to minimize a weighted sum of the Type-I and Type-II error, i.e. minimize:

$$C_0 \Pr(\hat{X}(Y) = 1|X = 0) + C_1 \Pr(\hat{X}(Y) = 0|X = 1)$$

for some $C_0 \ge 0$ and $C_1 \ge 0$, what test should we use?

3. **Gaussians and the MSE**

Suppose you draw n i.i.d. data points $(x_1, y_1), \ldots, (x_n, y_n)$, where n is a positive integer and the true relationship is $Y = WX + \varepsilon$, $\varepsilon \sim \mathcal{N}(0, \sigma^2)$. (That is, Y has a linear dependence on X, with additive Gaussian noise.) Show that finding the MLE estimate of W given the data points $\{(x_i, y_i) : i = 1, \ldots, n\}$ is equivalent to minimizing the cost function

$$J(w) = \sum_{i=1}^{n} (y_i - wx_i)^2$$