EECS 126 Probability and Random Processes — UC Berkeley, Fall 2019
Shyam Parekh — December 18, 2019

Final

Last Name	First Name	SID

Left Neighbor First and Last Name	**Right Neighbor** First and Last Name

Rules.

- **Unless otherwise stated, all your answers need to be justified.**
- No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.
- You have 10 minutes to read the exam and 160 minutes to complete it.
- The exam is not open book. No calculators or phones allowed.

Problem	points earned	out of
Problem 1		62
Problem 2		16
Problem 3		16
Problem 4		16
Problem 5		16
Total		126

1 Assorted Problems [62]

(a) **Tossing Coins [5]**

A fair coin is tossed eleven times. What is the probability that the sequence of outcomes is a palindrome (i.e. a sequence that is the same when reversed)?

(b) **Sketchy Bounds [5]**

As we derived in the homework, the element-wise expectation and variance of $\hat{I} = S^T S$ where S is a $d \times n$ Gaussian sketch matrix (i.e. $S_{ij} \overset{\text{iid}}{\sim} \mathcal{N}(0, \frac{1}{d})$) are

$$\mathrm{E}[\hat{I}_{ij}] = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}, \mathrm{Var}[\hat{I}_{ij}] = \begin{cases} 2/d, & \text{if } i = j \\ 1/d, & \text{otherwise} \end{cases}$$

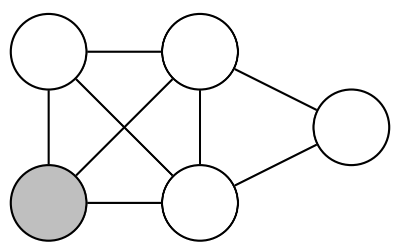
We want to reduce noise by increasing the dimension d of the sketching matrix. Using Chebyshev's inequality, find a lower bound on d in terms of ϵ such that for the diagonal entry $\hat{I}_{ii}$, we have the bound $P(\hat{I}_{ii} \in [1-\epsilon, 1+\epsilon]) \geq \frac{3}{4}$.

(c) **Exponential Shifting [5]**

Let X, Y be distributed independently as Exponential(λ), Exponential(μ). What is the probability that X is less than Y by at least some fixed amount $c \geq 0$?

(d) **Undirected Markov Chain [5]**

Consider a Markov chain defined on the following undirected graph. At each time step, you pick one of your neighbors (you cannot pick yourself) uniformly at random to move to. What is the stationary distribution probability of the shaded state?



(e) **Leaving [6]**

Suppose people enter a waiting room according to a Poisson Process with rate λ. Upon a new arrival, each person in the waiting room before the arrival leaves independently with probability $p < 1$. At time 0, the room is empty. At time T, what is the expected number of people in the waiting room?

Hint: Condition on the number of arrivals, N. The MGF of a Poisson random variable Z with rate λ is $M_Z(s) = e^{\lambda(e^s - 1)}$

(f) **Messages [5]**

Justin and Hong are continuously sending messages to you. Each of their messages arrive according to a Poisson Process, and their rates are λ_1 and λ_2, respectively. What is the expected amount of time, T, until you see a message from Justin directly followed by a message from Hong?

Example: For example you will record HJH if you get an arrival from Hong at time T_a, then from Justin at time T_b, then from Hong at time T_c, at which point you will have seen the pattern. Here, $T = T_c$.

(g) **Revisiting ER [5]**

Consider a set of N vertices of a graph. Each vertex i is associated with a $X_i \sim \mathcal{N}(0,1)$ RV, all i.i.d. Suppose we draw an edge between vertex i and vertex j if $X_i + X_j > c$ for some constant c.

(i) What is the probability that a particular edge exists? Your answers to the following questions may be expressed in terms of Φ, the standard Gaussian CDF.

(ii) Is this an ER random graph? Justify your answer.

◯ True ◯ False

(h) **Really Random Binomial MAP [5]**

Suppose you have a binomial variable $X \sim \text{Binomial}(n, U)$. If you know U is distributed as follows

$$U = \begin{cases} 0 \text{ w.p } 0.2 \\ 0.5 \text{ w.p } 0.6 \\ 1 \text{ w.p } 0.2 \end{cases}$$

what is the MAP estimate of U given X.

(i) **Two Sided Hypothesis Test [5]**

Suppose we observe n samples $X_1, X_2, \ldots, X_n$ where $X_i \sim \mathcal{N}(0, \sigma^2)$. We know

$$X = \begin{cases} 0 & \text{if } \sigma = \sigma_0 \\ 1 & \text{if } \sigma = \sigma_1 \end{cases}$$

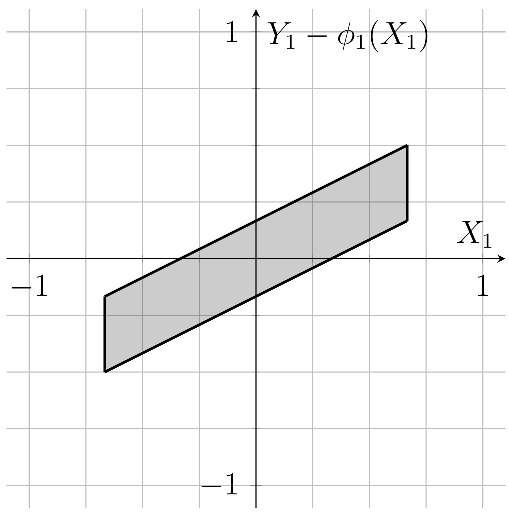
where $\sigma_0 < \sigma_1$. Show that the test $\hat{X}$ that maximizes $P(\hat{X} = 1 \mid X = 1)$ while ensuring $P(\hat{X} = 1 \mid X = 0)$ is at most β is of the form

$$\hat{X} = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} X_i^2 > c \\ 0 & \text{otherwise} \end{cases}$$

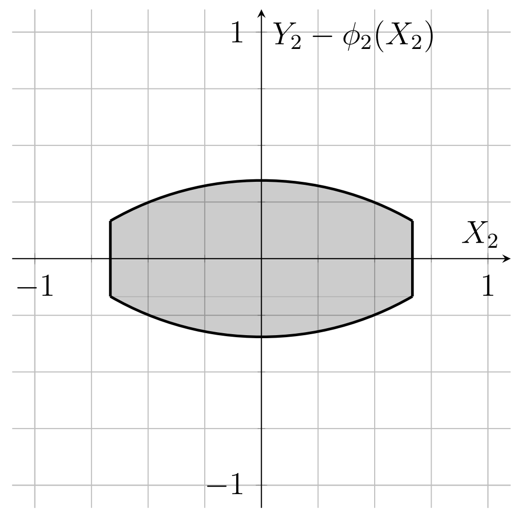
You do not need to find the specific value of c.

(j) **Graphical Estimators [6]**

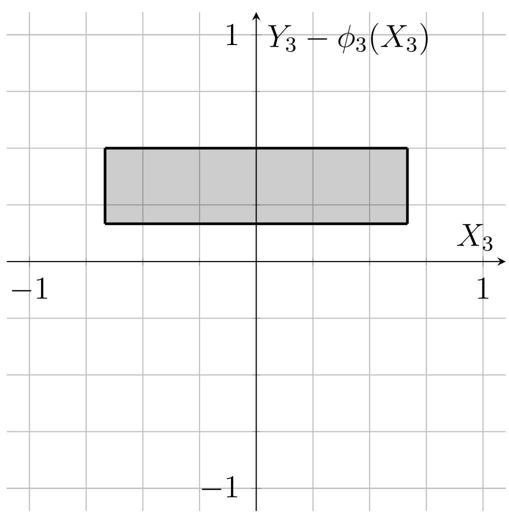
Each of the following 4 plots correspond to an estimator $\phi_i(X_i)$ of Y_i given X_i. In each plot, the joint density of $(X_i, Y_i - \phi_i(X_i))$ is shown. Assuming the density is the uniform distribution on the shaded area, could $\phi_i(X_i)$ be the LLSE and/or the MMSE? Use properties you know about the LLSE and MMSE. **No justification is necessary.**



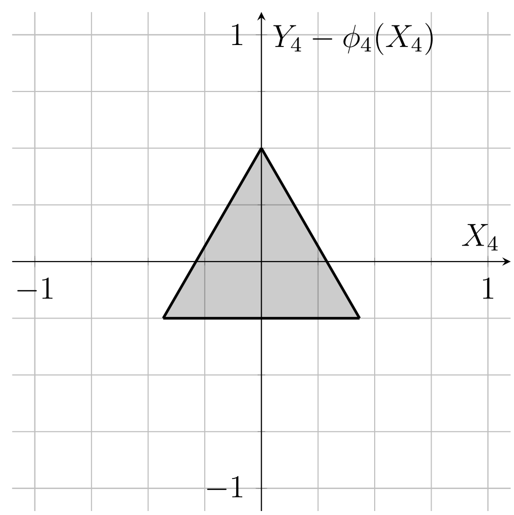
◯ Both ◯ MMSE ◯ LLSE ◯ None



◯ Both ◯ MMSE ◯ LLSE ◯ None



◯ Both ◯ MMSE ◯ LLSE ◯ None



◯ Both ◯ MMSE ◯ LLSE ◯ None

(k) **Jointly Gaussian Probability [5]**

Let X be distributed as $\mathcal{N}(0,1)$ and Y be distributed as $\mathcal{N}(1,1)$ with covariance 0.5. Define $W = X - Y$. Find $\Pr(W > Y)$ in terms of the standard Gaussian CDF, Φ.

(l) **Jointly Gaussian True or False [5]**

For the following two questions, justify your answer or describe a counterexample.

(i) If two random variables X and Y are marginally Gaussian, then they are jointly Gaussian.

◯ True ◯ False

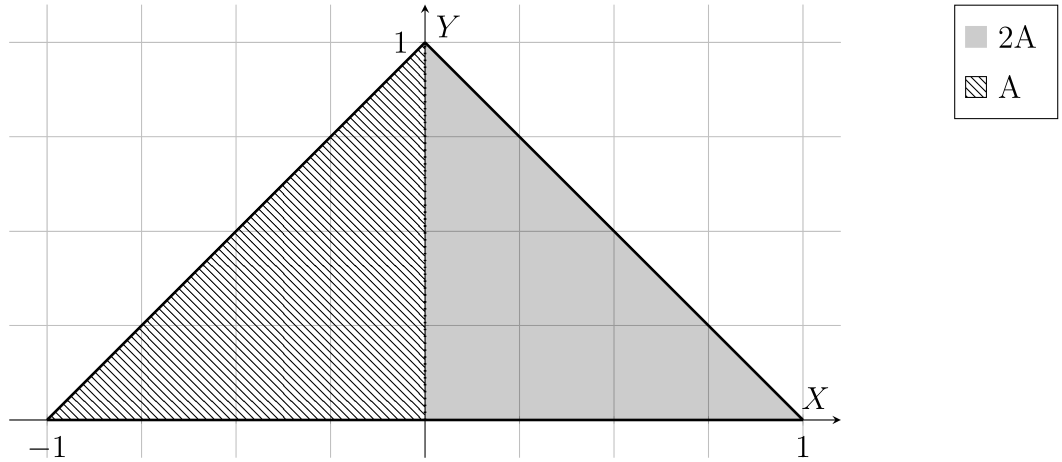
(ii) If two independent random variables X and Y are marginally Gaussian, then they are jointly Gaussian.

◯ True ◯ False

(iii) If $L[X \mid Y] = E[X \mid Y]$, then X and Y are jointly Gaussian.

◯ True ◯ False

2 Graphical Density [16]



(a) **MMSE [4]**

Find the MMSE of Y given X.

(b) **Covariance [6]**

Show that $\operatorname{cov}(X, Y) = -\frac{\operatorname{var}(Y)}{6}$.

(c) **LLSE [6]**

Find the LLSE of Y given X, $L[Y \mid X]$. Hint: To find $\operatorname{var}(X)$, think about the random variable $|X|$.

3 Operating Systems [16]

In an operating system, new tasks arrive to a queue according to a PP with rate λ. Furthermore, suppose tasks are processed one by one in the order they arrived, and that the processing time for each task independently has an exponential distribution with rate μ. For all of the parts, suppose $\lambda < \mu$ and that a long time has passed.

(a) **Expected Number of Tasks [6]**

What is the expected number of tasks $\mathrm{E}[X]$ in the queue?

(b) **Expected Delay [5]**

A new task arrives on the queue. What is the expected delay $\mathrm{E}[T]$ before it is done processing? You may write your answer in terms of your answer to the previous question, $\mathrm{E}[X]$.

(c) **LLSE [5]**

Suppose the level of the noise your computer fan makes depends on the number of tasks in the queue. In particular, the noise level $Z \sim \text{Poisson}(X)$. What is $L[X \mid Z]$? You may write your answer in terms of $\mathrm{E}[X]$ and $\text{var}(X)$.

4 Gaussian Poker [16]

Justin and Will are playing poker. Justin wants to estimate the true value of Will's hand from his bets. The game can be modeled as follows.

- Initially his hand has value $X_0 \sim \mathcal{N}(0, 3)$. It can actually be negative.
- Every round, a new card is drawn, and the value of his hand changes to $X_n = X_{n-1} + V_n$, where $V_n \sim \mathcal{N}(0, 1)$ $(n = 1, 2, \dots)$.
- After each round, he bets $Y_n = X_n + W_n$, where $W_n \sim \mathcal{N}(0, \sigma_w^2)$ $(n = 1, 2, \dots)$. These can also be negative.
- X_0, $V_1, V_2, \dots$, and $W_1, W_2, \dots$ are all independent.

(a) **First Round [5]**

Suppose $\sigma_w^2 = 2$. What is $E[X_1 \mid Y_1]$?

(b) **Second Round [6]**

Again suppose $\sigma_w^2 = 2$. What is $E[X_2 \mid Y_1, Y_2]$?

(c) **He's Bluffing [5]**

Suppose Justin didn't actually know Will's betting habits, but wanted to estimate his spread. Given a record of Will's bets $Y_1, \dots, Y_n$ and the value of his hand on the corresponding rounds $X_1, \dots, X_n$ find the MLE of the standard deviation σ_w.

5 Gaussian Process [16]

We define a **Gaussian Process** to be a sequence of random variables $X_1, X_2, \ldots$ such that any finite subset $X_{i_1}, X_{i_2}, \ldots, X_{i_n}$ is jointly Gaussian. (Thus, this further implies that each of the random variables X_i are Gaussian.) Suppose that the mean of each variable is $\mathbb{E}[X_i] = 0$, and the covariance between two of the Gaussians X_i, X_j is

$$\operatorname{cov}(X_i, X_j) = \frac{1}{ij} \min(i, j).$$

(a) **Covariance Matrix [5]**

Compute the covariance matrix of the random vector $\begin{bmatrix} X_i \\ X_j \end{bmatrix}$ where $i < j$.

(b) **Orthogonalization [6]**

For $i < j$, find the random vector $\begin{bmatrix} Y_i \\ Y_j \end{bmatrix}$ such that Y_i, Y_j are linear combinations $a_i X_i + b_i X_j, a_j X_i + b_j X_j$ of the random variables X_i, X_j and the covariance matrix of $\begin{bmatrix} Y_i \\ Y_j \end{bmatrix}$ is the identity matrix I.

(c) **Process Convergence [5]**

Prove that the sequence of random variables $\{X_i\}_{i=1}^{\infty}$ converges to 0 almost surely. Hint: Choose $i = j - 1$ in part (b) and see if you can rewrite the Gaussian Process $\{X_i\}_{i=1}^{\infty}$ so that we can apply SLLN.