Final Exam

Last Name	First Name	SID

Rules.

- No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.

- You have 10 minutes to read the exam and 170 minutes to complete it.

- The exam is not open book; we are giving you a cheat sheet. No calculators or phones allowed.

- Unless otherwise stated, all your answers need to be justified. Show all your work to get partial credit.

- Maximum you can score is 132 but 100 points is considered perfect.

Problem	points earned	out of
Problem 1		50
Problem 2		18
Problem 3		12
Problem 4		12
Problem 5		20
Problem 6		20
Total		132

Problem 1: Answer these questions briefly but clearly.

(a)[6] CLT True / False: No justification is required.

- $\lim_{n\to\infty} P(Binomial(n,p) > np) = p$ ○ True   ○ False

- $\lim_{n\to\infty} P(Poisson(n) > n) = \frac{1}{2}$ ○ True   ○ False

- $\lim_{n\to\infty} P(Exponential(n) > \frac{1}{n}) = \frac{1}{2}$ ○ True   ○ False

(b)[3] Order Statistic: Given that the 5th arrival time of a $Poisson(\lambda)$ process with $\lambda = 10$ occurs at time $t = 1$ second, what is the expected arrival time of the 2nd arrival?

(c)[3] MMSE Sanity Check: Assume that X and Y are two random variables such that $\mathbb{E}[X|Y] = L[X|Y]$. Then it must be that (choose the correct answers, if any):

○ X and Y are jointly Gaussian.

○ X can be written as $X = aY + Z$, where Z is a random variable independent of Y.

○ $\mathbb{E}((X - L[X|Y])Y^k) = 0 \quad \forall k \geq 0$

(d)[4] MMSE: Let X and Y be independent Gaussian random variables each with mean zero, and $Var(X) = \sigma_x^2$, $Var(Y) = \sigma_y^2$. Find $\mathbb{E}[X|e^{X+Y}]$.

(e)[5] Random Graph on a Random Graph: Suppose we generate a random graph by starting with an Erdos-Renyi graph $G(n,p)$. Then, we generate a random graph using the Erdos-Renyi model again on the subgraph of singletons (that is, each edge between two singletons is added with probability p). Calculate the expected number of edges in total.

(f)[3] MMSE: Given three i.i.d. random variables X, Y, Z, what is $\mathbb{E}[X|X+Y+Z]$?

(g)[3+3] Jointly Gaussian: Let X, Y and Z be jointly Gaussian random variables having covariance matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

and mean vector $\begin{bmatrix} 0 & 10 & 0 \end{bmatrix}^T$.

(i) Find $\mathbb{E}[Y|X, Z]$.

(ii) Find $\mathbb{E}[(e^X - e^{-X})Y(\sin Z)]$. (*Hint*: Condition on (X, Z).)

(h)[5] Neyman Pearson testing: Ray's posts on Piazza can be modeled as a Poisson process. Let its rate be λ_0 according to the null hypothesis H_0 and λ_1 according to the alternate hypothesis H_1, where $\lambda_1 > \lambda_0$. Say you observe the first post at time y_1. Describe the optimal Neyman Pearson (NP) hypothesis test for this problem. Assume the maximum probability of false alarm is ϵ, where $0 < \epsilon < 1$.

(i)[3] Huffman Tree:

Let X be a discrete random variable taking on 4 values A, B, C, and D. If we were to encode X with Huffman encoding, the resulting Huffman tree would look like this.
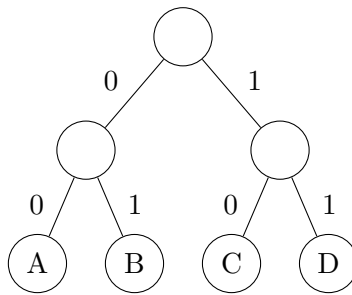
True or False: $H(X) \geq 1$? Justify.

(j)[3] Bipartite Markov Chain: Suppose in a bipartite graph you have two sets of nodes, L and R, of sizes m and n, such that for each $u \in L$ and $v \in R$, the transition probability from u to v is $1/n$, and the transition probability from v to u is $1/m$. Calculate the stationary distribution.

(Note: A bipartite graph has two sets of nodes where nodes in each set are only connected to the nodes in the other set.)
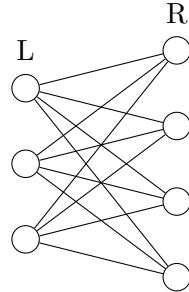
Complete Bipartite Graph for $m = 3, n = 4$

(k)[4] MLE with Numbered Balls: A box is filled with N balls numbered 1 through N. I randomly select K balls from the box. I order the balls in ascending order of their numbers and find them to be x_1, x_2,... x_K. What is the maximum likelihood estimate of N given my K observations? Justify your answer to get credit.

(1)[2+3] Fun with Gaussians:

(i) Show that the sum of two independent Gaussian random variables is Gaussian.

(ii) Show that the sum of two jointly Gaussian random variables is Gaussian, starting from the definition that two jointly Gaussian random variables X, Y can be written as linear combinations of underlying independent standard Gaussians Z_1, Z_2, i.e,

$$X = a_X Z_1 + b_X Z_2 + \mu_X$$

$$Y = a_Y Z_1 + b_Y Z_2 + \mu_Y$$

Problem 2 [2+5+4+7]: Graphical Density

Let X and Y have joint PDF as depicted below.
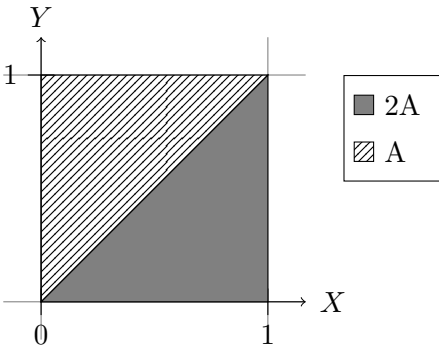

(a) Determine the value of A.

(b) Compute $E[X|Y]$.

(c) Compute $L[X+Y|X-Y]$.

(d) Compute $E[\max(X,Y)|\min(X,Y) \le 0.5]$.

Problem 3 [4+4+4]: Markov Gainz

Ray has an energy level $X_n \in \{0, 1, .., E\}$ units on the n-th day. Every day, with probability p, he takes a good rest which increases his energy level by 1 unit, and with probability q, he parties which decreases his energy level by 1 unit. Otherwise, the energy level remains the same. The energy level X_n can be described by the following birth-death chain:
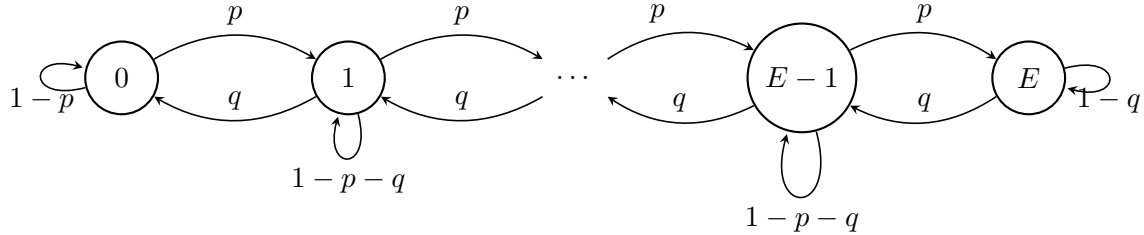
Ray goes to the gym every day, and does $Y_n = \begin{cases} \text{Poisson}(X_n) & \text{if } X_n > 0 \\ 0 & \text{if } X_n = 0 \end{cases}$ bench presses.

(a) Ray has an energy level of 0 units today. How many days on average will it take for him to have energy level of 2 units (assume that $E > 2$)?

(b) **For this part only**, assume that $E = \infty$, and $p/q \leq 1$. Suppose Ray has been going to the gym for a very long time. How many bench presses will Ray do today in expectation?

(c) Ray did 0 bench presses on the nth day (that is, $Y_n = 0$). Find the ratio p/q such that the posterior of X_n (that is, $P(X_n|Y_n = 0)$) is uniform over all energy levels $\{0, ..., E\}$. Assume that his prior on the energy levels (that is, $P(X_n)$) is the stationary distribution.

Problem 4 [3+5+4]: Fair and Loaded Coins

We have two indistinguishable coins, one fair and one loaded. The fair coin (F) has probability of heads 0.5 and the loaded coin (L) has probability of heads 1. We do n coin flips. For the first coin flip, we choose one of the F or L coins with equal probability. For every subsequent coin flip, the coin is chosen according to the following Markov chain:

$$\beta \circlearrowright \boxed{F} \quad \overset{1-\beta}{\underset{1-\alpha}{\rightleftarrows}} \quad \boxed{L} \circlearrowleft \alpha$$
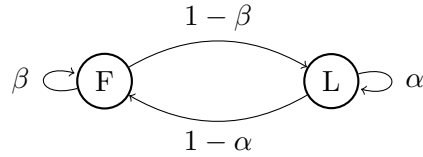
e.g., if on coin flip j, you flipped the F coin, on coin flip $j+1$, you have a β chance of flipping of the F coin and a $(1-\beta)$ chance of flipping the the L coin. We want to find the MLSE of the label of the coins given an observed Heads/Tails sequence.

(a) Suppose you observed T in the current state and H in the next state. Populate the one-stage trellis diagram shown below with appropriate costs (negative log-likelihoods as seen in lecture).
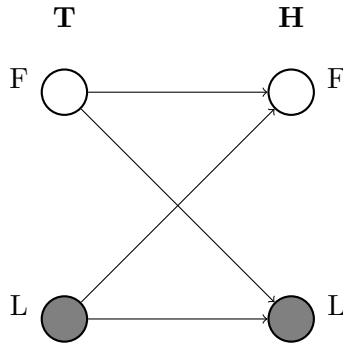

Figure 1: One "Stage" of the Trellis Diagram

(b) Given that $n = 3$, $\beta = 3/4$, $\alpha = 1/2$ and the sequence $\{\mathbf{T},\mathbf{H},\mathbf{H}\}$ is observed, draw the corresponding trellis diagram for estimating the sequence of which coin was used for each flip and write down the MLSE (maximum likelihood sequence estimate) of the label of coins. (Take $\log_2 3 = 1.6$ for easier calculations)

(c) For part (b), what is the MLE of the coin label for the second flip?

Problem 5 [3+3+3+6+5]: Kalman Filters and LLSE
Suppose we have the following dynamical system of equations:

$$X_n = \rho X_{n-1} + V_n, \ n = 2, 3, ... \ (\text{with } X_1 = V_1)$$

$$Y_n = X_n + W_n, \ n = 1, 2, ...$$

where V_n, and W_n for $n = 1, 2....$ are i.i.d $\mathcal{N}(0, 1)$ noise random variables and $|\rho| < 1$.

(a) What is the variance of X_n as $n \to \infty$?

(b) (i) Find $L[X_1|Y_1]$ geometrically through a vector space representation of the random variables X_1, Y_1, and W_1. Mark your plot clearly.

(ii) Find the expected mean-squared estimation error in estimating X_1 given Y_1.

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(c) Find the prediction estimate of X_2 given Y_1, i.e. $L[X_2|Y_1]$, as well as the expected mean-squared estimation error in estimating X_2 given Y_1.

(d) Now you want to update your estimate of X_2 given Y_1 and Y_2 by forming:

$$L[X_2|Y_1, Y_2] = L[X_2|Y_1] + L[X_2|\tilde{Y_2}]$$

(i) What is $\tilde{Y_2}$ in the above equation? Express it geometrically in terms of Y_1 and Y_2.

(ii) Find $L[X_2|\tilde{Y_2}]$ and the MMSE estimate of X_2 given Y_1 and Y_2.

(iii) What is the expected mean-squared-error in estimating X_2 given Y_1 and Y_2? How does it compare to the estimation error in part (c)?

(e) Now you want to further update your estimate of X_2 given Y_1, Y_2 and Y_3. Find $L[X_2|Y_1, Y_2, Y_3]$ and the expected mean-squared estimation error in estimating X_2 given Y_1, Y_2, and Y_3. How does this compare to the estimation error in parts 3 and 4?

Problem 6 [4+5+5+6]: Continuous Random Walk on a Grid

An ant performs a continuous time random walk on the non-negative integer lattice. At any time $t \geq 0$, the position of the ant $Z(t)$ is a tuple $(X(t), Y(t))$. The ant starts in state $(0,0)$. At any time, the ant moves to the right with rate λ and up also with rate λ, so that the position of the ant is described by an infinite CTMC on the state space $\mathbb{N} \times \mathbb{N}$, as pictured below.
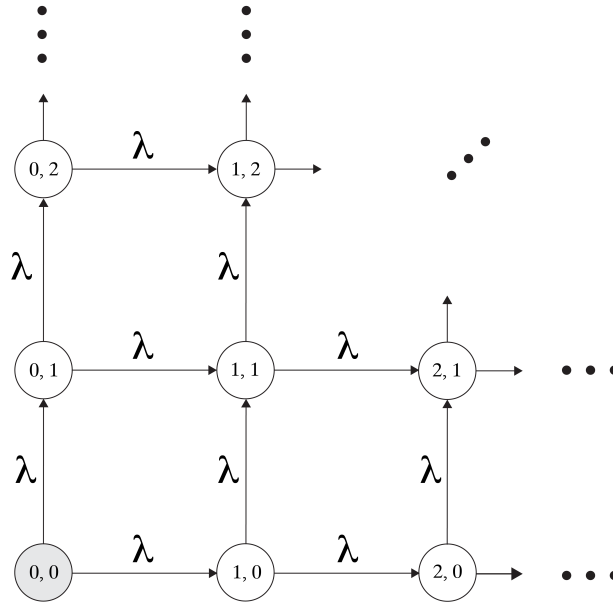
(a) Argue that $X(t)$ and $Y(t)$ are independent Poisson Processes and write down their rates.

(b) At time $t = 1$, the ant is at position $(3, 1)$. What is the probability that at time $t = 0.75$ the ant was at position $(3, 0)$?

(c) Denote by V_n the ant's average speed at time $t = n$, that is, $V_n = (X(n) + Y(n))/n$. Does the sequence $(V_n)_{i=1}^{\infty}$ converge a.s? If not, prove it. If yes, specify what it converges to and justify (assume n to be an integer).

(d) Now, modify the walk by allowing the ant to also move left (if possible) at rate μ and down (if possible) also with rate μ. The new CTMC is pictured below (all down and left arrows have rate μ and all up and right arrows have rate λ). For $\lambda < \mu$, find the stationary distribution of the corresponding CTMC (Hint: Use a symmetry argument that parallels your argument in part (a)).