Midterm 1

Last Name	First Name	SID

- You have 10 minutes to read the exam and 90 minutes to complete this exam.

- The maximum you can score is 120, but 100 points is considered perfect.

- The exam is not open book, but you are allowed to consult the cheat sheet that we provide. No calculators or phones. No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.

- Show all work to get any partial credit.

- Take into account the points that may be earned for each problem when splitting your time between the problems.

Problem	points earned	out of
Problem 1		40
Problem 2		20
Problem 3		20
Problem 4		20
Problem 5		20
Total		120

Problem 1: Answer these questions briefly but clearly. [40]

(a) Covariance and Independence [5]

Exhibit a pair of random variables (X, Y) such that X and Y are dependent, but $\text{cov}(X, Y) = 0$

(b) MGF [5] Let $M_X(s)$ be the moment generating function of a random variable X. Which of the following are valid moment generating functions? If valid, prove which random variable (as a function of X) the MGF belongs to. If invalid, justify.

1. $M_X(s)M_X(2s)$

2. $2M_X(s)$

3. $e^{-2s}M_X(s)$

(c): Book Sale [10] Bob is at a book sale. There are T books in all and L books that he likes. Bob picks up a book at random and buys it if he likes it. Books that are considered once are not considered again. Let X be the number of books he must examine to find n books that he likes. (Here, $n, L, T \in \mathbb{N}$ are fixed numbers ($n \leq L \leq T$), while X is a random variable. The final answers should not involve summations but can have expressions like $\binom{n}{k-1}$, as well as factorials.)

1. Find $E[X]$. (Hint: You do not require the density to find this. Think about symmetry and then use linearity).

2. Find $P(X = x)$. (Hint: Find the right quantity to condition on).

(d) Points in square [**5**] Two points are placed uniformly at random and independently in $[0,1]^2$. What is the expected value of the square of the distance between the two points?

(e): Light bulbs [**5**] A lighting company tests 10 bulbs by turning them on at the same time. Each bulb has a lifetime exponentially distributed with parameter λ. Let X be the length of the time interval between the time that the first bulb burns out and the second bulb burns out. Find $f_X(x)$.

(f) Fountain Codes [10]

100 data chunks need to be transmitted over a packet erasure channel (as in Lab 2). We will encode the data chunks into 200 packets using the following scheme. For each packet, we will first roll a 6-sided fair die, and based on the outcome, sample uniformly at random without replacement that many of the 100 data chunks and XOR the data chunks. (As an example, if the die comes up "3", then the corresponding packet will be the XOR of three uniformly sampled random data chunks from 1 to 100, say data chunks 4, 43 and 87).

1. What is the probability that a data chunk is connected to a given packet?

2. What is the average number of packets that a data chunk will be associated with?

3. What is the probability that a data chunk is not connected to any packet?

Problem 2: Magnets [20]

There are n bar magnets, $n > 1$, placed in a line end to end. Assume that each magnet takes one of the two possible orientations, say (NS) or (SN), with equal probability, and magnets have independent orientations. Adjacent magnets with like poles repel, while those with opposite poles join and form blocks. For instance, if $n = 5$, and the orientation of magnets is $(NS)(SN)(SN)(NS)(NS)$, they form 3 blocks of the form (NS) | $(SN)(SN)$ | $(NS)(NS)$. Let N be the number of blocks of joint magnets.

1. What is $\mathbb{E}(N)$?

2. What is $\mathrm{Var}(N)$?

Problem 3: Squared sum of Gaussians [20]

Let X and Y be independent Gaussian random variables with mean 0 and standard deviation 1.

1. Derive the probability density function (pdf) of $X^2 + Y^2$.

2. For $t > 2$, provide upper bounds on $\mathbb{P}(X^2 + Y^2 > t)$ using Markov and Chebyshev inequality.

Problem 4: Graphical Density [20]

Let (X, Y) be uniformly distributed over the triangle with vertices $(0,0)$, $(1,0)$, and $(2,1)$.

1. Find $f_{X,Y}(x, y)$ and $f_X(x)$.

2. Compute $\mathbb{E}[Y \mid X = x]$.

Problem 5: Sum of Poisson Squares [20] Let $X_i, i \geq 1$ be i.i.d. Poisson random variables with parameter λ. Also, let N be a geometric random variable with parameter p which is independent from all X_i's. With this, define $S := X_1^2 + X_2^2 + \cdots + X_N^2$.

1. Find $\mathbb{E}(S)$.

2. For an integer $k \geq 1$, find $\mathbb{E}(S|N > k)$.