UC Berkeley
Department of Electrical Engineering and Computer Sciences

EECS 126: Probability and Random Processes

Problem Set 10 (Optional)
Spring 2021

1. **Frogs**

   Three frogs are playing near a pond. When they are in the sun they get too hot and jump in the lake at rate 1. When they are in the lake they get too cold and jump onto the land at rate 2. The rates here refer to the rate in exponential distribution. Let X_t be the number of frogs in the sun at time $t \geq 0$.

   (a) Find the stationary distribution for $(X_t)_{t \geq 0}$.

   (b) Check the answer to (a) by noting that the three frogs are independent two-state Markov chains.

2. **Taxi Queue**

   Empty taxis pass by a street corner according to a Poisson process of rate two per minute and pick up a passenger if one is waiting there. Passengers arrive at the street corner according to a Poisson process of rate one per minute and wait for a taxi only if there are less than four persons waiting; otherwise they leave and never return. John arrives at the street corner at a given time. Find his expected waiting time, given that he joins the queue. Assume that the process is in steady state.

3. $M/M/2$ **Queue**

   A queue has Poisson arrivals with rate λ. It has two servers that work in parallel. When there are at least two customers in the queue, two are being served. When there is only one customer, only one server is active. The service times are i.i.d. exponential random variables with rate μ. Let $X(t)$ be the number of customers either in the queue or in service at time t.

   (a) Argue that the process $(X(t), t \geq 0)$ is a Markov process.

   (b) Draw the state transition diagram.

   (c) Find the range of values of μ for which the Markov chain is positive-recurrent and for this range of values calculate the stationary distribution of the Markov chain.

4. **Reversibility of CTMCs**

   We say a CTMC with rate matrix Q is *reversible* if there is a distribution p satisfying the detailed balance equations:
   $$p_i q_{ij} = p_j q_{ji} \quad \forall i, j.$$
   Show that if a CTMC is reversible w.r.t. p, then p is a stationary distribution for the chain. Furthermore, show that in this case the embedded chain is also reversible. *Remark.* The converse is true too, i.e. the CTMC is reversible given that the embedded chain is reversible.

5. Connected Random Graph

We start with the empty graph on n vertices, and iteratively we keep on adding undirected edges $\{u, v\}$ uniformly at random from the edges that are not so far present in the graph, until the graph is connected. Let X be a random variable which is equal to the total number of edges of the graph. Show that $\mathbb{E}[X] = O(n \log n)$.

Hint: consider the random variable X_k which is equal to the number of edges added while there are k connected components, until there are $k - 1$ connected components. Don't try to calculate $\mathbb{E}[X_k]$, an upper bound is enough.